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ORIGINAL RESEARCH

A Two–parameter Family of Exponentially–fitted Obrechkoff Methods for Second-order Boundary Value Problems

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Abstract:
Introduction: Generally, classical numerical methods may not be well suited for problems with oscillatory or periodic behaviour. To overcome this deficiency, they are modified using a technique called exponential fittings. The modification makes it possible to construct new methods suitable for the efficient integration of oscillatory or periodic problems from classical ones.
Aims: In this work, a two–parameter family of exponentially–fitted Obrechkoff methods for approaching problems that exhibit oscillatory or periodic behaviour is constructed.
Materials and Methods: The construction is based on a six-step flowchart described in the literature.
Results: Unlike the single–frequency method in the literature, the constructed methods depend upon two frequencies which can be tuned to solve the problem at hand more accurately. The leading term of the local truncation error of the new family of method can also be easily obtained from the given general expression. The efficiency of the new methods is demonstrated on some numerical examples
Conclusion: This work provides extension to the results obtained in by authors in the literature.
Keywords: Multiparameter, Exponentially–fitted, Obrechkoff Method, Oscillatory, Periodic

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1 INTRODUCTION

In this paper, the second–order two–point boundary value problem of the form

$$u'' = f(t,u), \;\; u(a) = \eta_1, \; u(b) = \eta_2 \tag{1}$$

with oscillatory or periodic behaviour is considered. Recently, the construction of methods that efficiently approach the oscillatory or periodic behaviour of the problem (1) is gaining popularity. Since classical methods may not be well suited for this purpose, they have to be adapted. This adaptation is the core of exponential fitting technique [13] where the adapted numerical method is developed in order to be exact on problems whose solution is linear combination of

$$1, t, \cdots, t^K, \exp(\pm\omega t), t\exp(\pm\omega t), \cdots, t^P\exp(\pm\omega t) \tag{2}$$

where *K* and *P* are integers. Several exponentially–fitted methods have been constructed and many classical methods have been adapted for problems with the fitting space (2). In [20], the exponentially–fitted variants of the classical two step Numerov method was constructed. The authors used the constructed variants to compute large number of eigenvalues of regular Sturm–Liouville problems and showed that the pure exponential version gave better accuracy. The authors in [11] gave the optimal exponentially fitted two-step Numerov method for solving two-point boundary value problems. More recently, in [9], hybrid two-step exponentially fitted methods were constructed and implemented on initial value problems. The author in [10] constructed the exponentially fitted versions of the classical two-step Numerov method and also constructed methods that are fitted exponentially for the solution of fourth-order boundary value problems. For the first time, the combination of exponential fitting and methods of Störmer/Verlet and Obrechkoff type were discussed by in [19] and [21] respectively. The authors in [19] constructed the exponentially-fitted variants of the Störmer/Verlet method for second order problems of the form $y'' = f(y)$. Exponentially-fitted Obrechkoff method with *m=2* was constructed for the problem $y'' = f(x,y)$ and the result for *m=3* was also stated. Authors in [21] also discussed the linear stability properties of the methods constructed. However, not so much has been done in the construction of exponentially fitted methods for fitting space with multi-pair frequencies. In [4, 5, 6], the authors constructed Runge–Kutta type method for the fitting space

$$1, t, \cdots, t^K, \exp(\pm\omega t), \exp(\pm 2\omega t), \cdots, \exp(\pm(P+1)\omega t). \tag{3}$$

The authors in [22] proposed a multiparameter exponentially-fitted Numerov method for the space

$$1, t, \cdots, t^K, \exp(\pm\omega_0 t), \exp(\pm\omega_1 t), \cdots, \exp(\pm\omega_p t). \tag{4}$$

for solving periodic problems with more than one frequency. In this paper, the general fitting space

$$\left.\begin{array}{r} 1, t, \cdots, t^K, \exp(\pm\omega_i t), t\exp(\pm\omega_i t), \cdots, t^{P_i}\exp(\pm\omega_i t), \\ i = 1, 2 \end{array}\right\} \tag{5}$$

is considered. This work is related to [20, 21] and shall provide extension to the results obtained in [21].

2 CONSTRUCTION OF METHOD

The classical symmetric Obrechhoff method for solving (1) is given by

$$u_{n+1} - 2u_n + u_{n-1} = \sum_{i=1}^{m} h^{2i}\left(\gamma_i u_{n+1}^{(2i)} + 2\beta_i u_n^{(2i)} + \gamma_i u_{n-1}^{(2i)}\right). \tag{6}$$

In this paper however, we consider the case of *m=2*. To construct the exponentially–fitted variants of (6), we rewrite (6) in a more general form as

$$u_{n+1} - 2\delta_0 u_n + u_{n-1} = \sum_{i=1}^{m} h^{2i}\left(\gamma_i u_{n+1}^{(2i)} + 2\beta_i u_n^{(2i)} + \gamma_i u_{n-1}^{(2i)}\right) \tag{7}$$

and follow the six step procedure described in ([13]) with a slight modification. Following step one of the procedure, the corresponding linear difference operator $\mathcal{L}[h,\gamma]$ reads

$$\mathcal{L}[h,\gamma]u(t) = u(t+h) - 2\delta_0 u(t) + u(t-h) - \sum_{i=1}^{m} h^{2i}\left(\gamma_i u^{(2i)}(t+h) + 2\beta_i u^{(2i)}(t) + \gamma_i u^{(2i)}(t-h)\right) \tag{8}$$

where $\gamma := (\delta_0, \gamma_1, \gamma_2, \beta_1, \beta_2)$. To compute the moments $L_k^*(\gamma) = h^k\mathcal{L}[h,\gamma]t^k\mid_{t=0}$, we apply step two of the procedure to obtain

$$\begin{aligned}
L_0^*(\gamma) &:= 2 - 2\delta_0 = 0\\
L_2^*(\gamma) &:= 2 - 4\beta_1 - 4\gamma_1 = 0\\
L_4^*(\gamma) &:= 2 - 48\beta_2 - 24\gamma_1 - 48\gamma_2 = 0\\
L_6^*(\gamma) &:= 2 - 60\gamma_1 - 720\gamma_2 = 0\\
L_8^*(\gamma) &:= 2 - 112\gamma_1 - 3360\gamma_2 = 0
\end{aligned}$$

Due to the symmetry of the method, $L_{2k+1}^*(\gamma) = 0$ for all $k \in \mathbb{N}$. The algebraic system above is compatible and one finds $M = 10$ (*i.e the maximal M for which a solution exists for the above system is 10*). Solving the above system one obtains

$$\left.\beta_1 = \tfrac{115}{252}, \quad \beta_2 = \tfrac{313}{15120}, \quad \gamma_1 = \tfrac{11}{252}, \quad \gamma_2 = -\tfrac{13}{15120}, \quad \delta_0 = 1 \;\right\} \tag{9}$$

The resulting classical method is

$$u_{n+1} - 2u_n + u_{n-1} = \frac{h^2}{252}\left(11u^{(2)}(t+h) + 230u^{(2)}(t) + 11u^{(2)}(t-h)\right) - \frac{h^4}{15120}\left(13u^{(2)}(t+h) - 616u^{(2)}(t) + 13u^{(2)}(t-h)\right) \tag{10}$$

We shall refer to the method (10) as $S0$. To construct the two-parameter exponentially-fitted variants of (10) with a reference set (in this case $I = 2$) of *M* functions (5), one applies the third step of the six-step procedure which results due to the symmetry in $G^-(\omega_{h_i},\gamma) = 0$ and

$$-\frac{1}{2}G^+(\omega_{h_i},\gamma) = \delta_0 + \beta_2\omega_{h_i}^4 + \beta_1\omega_{h_i}^2 + \left(\gamma_2\omega_{h_i}^4 + \gamma_1\omega_{h_i}^2 - 1\right)\cosh(\omega_{h_i}) \tag{11}$$

where $\omega_{h_i} = \omega_i h$. Since $M = 10$, step four of the procedure gave rise to a two–parameter family of six exponentially fitted methods characterized by:

$S1: (K, P_1, P_2) = (-1, 0, 3)$: The two-parameter exponentially fitted case with the set

$$\{\exp(\pm\omega_1 t), \exp(\pm\omega_2 t), t\exp(\pm\omega_2 t), t^2\exp(\pm\omega_2 t), t^3\exp(\pm\omega_2 t)\}$$

$S2: (K, P_1, P_2) = (-1, 1, 2)$: The two-parameter exponentially fitted case with the set

$$\{exp(\pm\omega_1 t), t\exp(\pm\omega_1 t), \exp(\pm\omega_2 t), t\exp(\pm\omega_2 t), t^2\exp(\pm\omega_2 t)\}$$

$S3: (K, P_1, P_2) = (1, 0, 2)$: The two-parameter exponentially-fitted case with the set

$$\{1, t, \exp(\pm\omega_1 t), \exp(\pm\omega_2 t), t\exp(\pm\omega_2 t), t^2\exp(\pm\omega_2 t)\}$$

$S4: (K, P_1, P_2) = (1, 1, 1)$: The two-parameter exponentially-fitted case with the set

$$\{1, t, \exp(\pm\omega_1 t), t\exp(\pm\omega_1 t), \exp(\pm\omega_2 t), t\exp(\pm\omega_2 t)\}$$

$S5: (K, P_1, P_2) = (3, 0, 1)$: The two-parameter exponentially-fitted case with the set

$$\{1, t, t^2, t^3, \exp(\pm\omega_1 t), \exp(\pm\omega_2 t), t\exp(\pm\omega_2 t)\}$$

$S6: (K, P_1, P_2) = (5, 0, 0)$: The two-parameter exponentially-fitted case with the set

$$\{1, t, t^2, t^3, t^4, t^5, \exp(\pm\omega_1 t), \exp(\pm\omega_2 t)\}$$

The coefficients for each case above are obtained by solving for γ, the nonlinear algebraic system

$$\begin{cases} L_k^*(\gamma), \quad k = 0, \cdots, K \\ G^{\pm\,(p)}(\Omega_{h_i}, \gamma), \quad 0 \le p \le P_i, \quad i = 1, 2 \end{cases}$$

In this paper, we shall concern ourself with only the case: $S1: (K, P_1, P_2) = (-1, 0, 3)$. The expressions for the coefficients of the case considered are obtained in series form as follows:

- Case 1 :: $S1: (K, P_1, P_2) = (-1, 0, 3)$:

$$\begin{aligned} \delta_0 &= \left(\frac{47441\omega_{h_2}^{10}}{86502659443200} - \frac{233\omega_{h_2}^{8}}{42247941120}\right)\omega_{h_1}^4 + \\ &\quad \left(\frac{59\omega_{h_2}^{8}}{152409600} - \frac{233\omega_{h_2}^{10}}{10561985280}\right)\omega_{h_1}^2 + 1 \end{aligned} \tag{12}$$

$$\begin{aligned} \gamma_1 &= \left(-\frac{23203\omega_{h_2}^{4}}{344044668240} + \frac{2357\omega_{h_2}^{2}}{15017822820} + \frac{233}{88016544}\right)\omega_{h_1}^4 \\ &\quad + \left(\frac{69556\omega_{h_2}^{4}}{18772278525} - \frac{52\omega_{h_2}^{2}}{2750517} - \frac{59}{317520}\right)\omega_{h_1}^2 \\ &\quad - \frac{59\omega_{h_2}^{2}}{79380} - \frac{391\omega_{h_2}^{4}}{22004136} + \frac{11}{252} \end{aligned} \tag{13}$$

$$\begin{aligned} \gamma_2 &= \left(-\frac{6017021\omega_{h_2}^{4}}{3633111696614400} + \frac{44753\omega_{h_2}^{2}}{1441710990720} - \frac{233}{1056198528}\right)\omega_{h_1}^4 \\ &\quad + \left(\frac{184763\omega_{h_2}^{4}}{2621292710400} - \frac{20087\omega_{h_2}^{2}}{13202481600} + \frac{59}{3810240}\right)\omega_{h_1}^2 \\ &\quad + \frac{59\omega_{h_2}^{2}}{952560} - \frac{83561\omega_{h_2}^{4}}{26404963200} - \frac{13}{15120} \end{aligned} \tag{14}$$

$$\begin{aligned} \beta_1 &= \left(\frac{23203\omega_{h_2}^{4}}{344044668240} - \frac{2357\omega_{h_2}^{2}}{15017822820} - \frac{233}{88016544}\right)\omega_{h_1}^4 \\ &\quad + \left(-\frac{69556\omega_{h_2}^{4}}{18772278525} + \frac{52\omega_{h_2}^{2}}{2750517} + \frac{59}{317520}\right)\omega_{h_1}^2 \\ &\quad + \frac{391\omega_{h_2}^{4}}{22004136} + \frac{59\omega_{h_2}^{2}}{79380} + \frac{115}{252} \end{aligned} \tag{15}$$

$$\begin{aligned} \beta_2 &= \left(\frac{8307851\omega_{h_2}^{4}}{3633111696614400} - \frac{157889\omega_{h_2}^{2}}{1441710990720} - \frac{1165}{1056198528}\right)\omega_{h_1}^4 \\ &\quad + \left(\frac{2304301\omega_{h_2}^{4}}{5766843962880} + \frac{144887\omega_{h_2}^{2}}{13202481600} + \frac{59}{762048}\right)\omega_{h_1}^2 \\ &\quad + \frac{318161\omega_{h_2}^{4}}{26404963200} + \frac{59\omega_{h_2}^{2}}{190512} + \frac{313}{15120} \end{aligned} \tag{16}$$

2.1 Local Truncation Error (LTE)

The general expression of the leading term of the local truncation error (*lte*) for an exponentially fitted method obtained in this way with respect to the basis (5) takes the form

$$\begin{aligned} lte^{EF}(t) &= (-1)^{\sum_{i=1}^{I} P_i + I} h^M \frac{\mathcal{L}_{K+1}^*(\gamma(\Omega_{h_i}))}{(K+1)!\Omega_{h_1}^{P_1+1}\cdots\Omega_{h_I}^{P_I+1}} \times \\ &\quad \prod_{i=1}^{I}(D^2 - \omega_i^2)^{P_i+1}, \quad D^m := \frac{d^m}{dt^m} \end{aligned} \tag{17}$$

with $K, P_1, \cdots, P_I$ and *M* satisfying the condition $K + 2(P_1 + \cdots + P_I) = M - 2I - 1$, ([2]). Using (17), the leading terms of the local truncation error for the case considered above is obtained as:

$$\begin{aligned} lte_{(-1,0,3)} = (\delta_0 - 1)\Bigg(&\frac{2u^{(10)}(t)}{\omega_1^2\omega_2^8} - \frac{2\left(\omega_1^2 + 4\omega_2^2\right)u^{(8)}(t)}{\omega_1^2\omega_2^8} - \\ &\frac{2\left(-6\omega_2^4 - 4\omega_1^2\omega_2^2\right)u^{(6)}(t)}{\omega_1^2\omega_2^8} - \frac{2\left(4\omega_2^6 + 6\omega_1^2\omega_2^4\right)u^{(4)}(t)}{\omega_1^2\omega_2^8} \\ &- \frac{2\left(-\omega_2^8 - 4\omega_1^2\omega_2^6\right)u''(t)}{\omega_1^2\omega_2^8} - 2\Bigg) \end{aligned} \tag{18}$$

3 NUMERICAL EXPERIMENTS

The aim of the computational analysis carried out in this section is to investigate the accuracy and efficiency of the constructed methods on some standard problems compared with some existing methods. Here focus is on two test cases. The first problem studied in [22] has frequencies which are neither real nor pure imaginary but rather are complex conjugates.

3.1 Problem 1

Consider the boundary value problem

$$y'' = \frac{3}{4}y - \exp(t)\sin\left(\frac{t}{2}\right), \quad y(0) = 1, \; y(\pi) = 0 \qquad (19)$$

whose solution is given by

$$y(t) = \exp(t)\cos\left(\frac{t}{2}\right) \qquad (20)$$

Problem (19) has two complex conjugate frequencies which are: $\omega_1 = 1 + \frac{1}{2}i$ and $\omega_2 = 1 - \frac{1}{2}i$. To investigate the accuracy of the constructed methods, Problem (19) is solved using one of the methods constructed in this work viz:$S1$: $(K, P_1, P_2) = (-1, 0, 3)$ and the results obtained are compared with the classical Obrechkoff method with M=2 (*cObrechkoff M=2*) and exponentially–fitted Multifrequency Numerov (M2Km1P4) of [22]. Using different values of steplength *h*, the maximum absolute value for each steplength is obtained as presented in Figure 1.

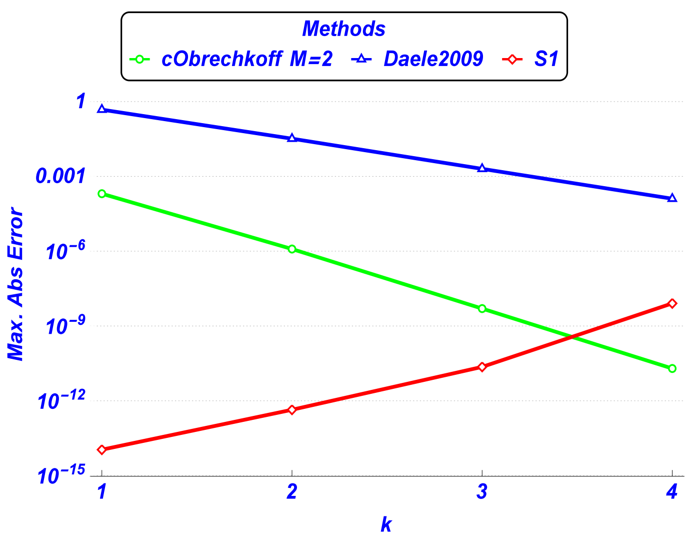

Figure 1: Maximum absolute errors for Problem 1 as a function of the step-size $h = 2^{-k}, k = 2(1)4$

3.2 Problem 2

The second problem considered in this paper is the boundary value problem

$$y'' = y + 2\exp(t) - 8\exp(3t), \; y(0) = -1, \; y(1) = \exp(1) - \exp(3) \quad (21)$$

with exact solution

$$y(t) = t\exp(t) - \exp(3t) \qquad (22)$$

As seen from Figure 2 the method also gave better results.

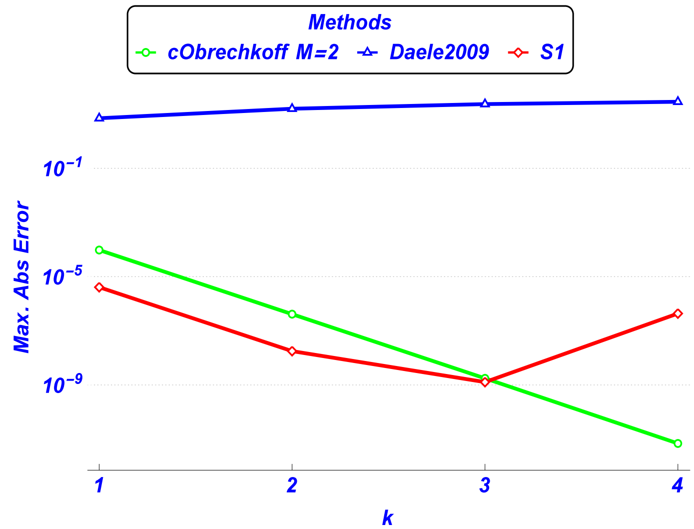

Figure 2: Maximum absolute errors for Problem 2 as a function of the step-size $h = 2^{-k}, k = 1(1)4$

4 CONCLUSION

In this paper, the work of the authors in ([21]) has been extended to allow for a larger set of fitting space for exponentially–fitted Obrechkoff methods. The extension involves the construction of a family of exponentially–fitted Obrechkoff methods suitable for the integration of periodic/oscilatory problems with two frequencies. The step by step application of the six-step procedure for the construction of the family of two–parameter exponentially fitted Obreckhoff methods has been presented. Unlike the single parameter method constructed in ([21]), the coefficients are now functions of two frequencies. The leading term of the local truncation error of the new family of method can also be easily obtained from the given general expression.

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