Research Article
Journal of Research and Review in Science, 78–81
Volume 4, December 2017

ORIGINAL RESEARCH

Stability and Convergence of Two-Step Obrechkoff Scheme For Second-Order Two-Point Boundary Value Problem

Bosede Olufemi Alfred, **Wusu Ashiribo Senapon** and **Akanbi Moses Adebowale**

Department of Mathematics, Lagos State University, Ojo. Lagos, Nigeria.

Correspondence
Bosede Olufemi Alfred, Department of Mathematics, Lagos State University, Ojo. Lagos, Nigeria.
Email:aolubosede@yahoo.co.uk

Funding information
This research work is self sponsored.

Abstract:
Introduction: Mathematical modeling of scientific and engineering processes often yield Boundary Value Problems (BVPs). One of the broad categories of numerical methods for solving Boundary Value Problems (BVPs) is the finite difference methods, in which the differential equation is replaced by a set of difference equations which are solved by direct or iterative methods.
Aims: This research focus on the establishment of conditions that ensure the stability and convergence of the two-step Obrechkoff method for solving $u'' = f(t,u)$, $a < x < b$, $u(a) = \eta_1$, $u(b) = \eta_2$.
Materials and Methods: In this paper, we use some properties of matrices to analyze the stability and convergence of the prominent finite difference methods - two-step Obrechkoff method - for solving the boundary value problem $u'' = f(t,u)$, $a < x < b$, $u(a) = \eta_1$, $u(b) = \eta_2$.
Results: Necessary conditions for the two-step Obrechkoff method to be convergent using the properties of matrices has been established. It has also been shown that the method is not P-stable but has a large interval of periodicity.
Conclusion: The necessary conditions for the two-step Obrechkoff method to be convergent using the properties of matrices has been established. It has also been shown that the method is not P-stable but has a large interval of periodicity.
Keywords: Convergence, Stability, Boundary Value Problem, Obrechkoff, Finite Difference Scheme

All co-authors agreed to have their names listed as authors.

1 INTRODUCTION

Numerical methods for solving boundary value problems can broadly be categorised into the following: *(i)* finite element methods, *(ii)* finite difference methods, *(iii)* shooting methods. Amongst these categories, the finite difference methods are widely used for the direct integration of the given problem. Finite difference methods are linear multistep methods that can be implicit or explicit [1]. The simplest of these category of methods is the Cowell's method of order two. The stability behavior of the Cowell's method when applied to the second-order initial value problem $u'' = f(t,u)$, $u(a) = u_0$, $u'(a) = u'_0$ has been studied by the authors in [2, 3]. Although there exist several Runge-Kutta type method for integrating second-order differential equation [4], there strengths lie in their application to initial value problems. Obrechkoff-type methods apparently are well-suited for both initial and boundary value problems. The simplest of the Obrechkoff-type method is the well-known Numerov method [5, 6, 1]. Authors in the past have constructed several Obrechkoff-type methods with higher orders for second order initial value problems [8, 9]. In the work of [10], the asymptotic stability of linear multistep methods for the direct integration of second-order problems are compared with those of the methods for integrating the corresponding system of first order equations. The P-Stability for the Obreckhoff methods with *m=2,3* when applied to second order initial value problems were discussed by the authors in [9]. In this work, using some properties of matrices, we established the conditions that will ensure the stability and convergence of the two-step Obrechkoff method when applied to the boundary value problem

$$u'' = f(t,u), \quad a < x < b, \quad u(a) = \eta_1, u(b) = \eta_2 \tag{1}$$

2 MATERIAL AND METHODS

2.1 Some Properties of Matrices

Definition 2.1 *A matrix* **A** $= (a_{ij})$ *is tridiagonal if* $a_{ij} = 0$, *whenever* $|i-j| > 1$.

Definition 2.2 *A tridiagonal matrix* **A** $= (a_{ij})$, *is irreducible if and only if* $a_{i,i-1} \neq 0,\ i = 2,3,\cdot,N$ *and* $a_{i,i+1} \neq 0,\ i = 1,2,\cdots,N-1$

Definition 2.3 *A tridiagonal matrix* **A** $= (a_{ij})$, *is diagonally dominant if*

$$|a_{ii}| = \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|, \quad i = 1,2,\cdot,N$$

Definition 2.4 *A matrix* **A** $= (a_{ij})$, *is said to be irreducibly diagonally dominant, if it is irreducible and diagonally dominant with inequality being satisfied for at least one* i.

Theorem 2.5 *A matrix* **A** $= (a_{ij})$, *is monotone if* **Az** $\geq$ **0** $\Rightarrow$ **z** $\geq$ **0**.

The main properties of a monotone matrix are as follows:

- The monotone matrix **A** is nonsingular
- A matrix **A** is monotone if and only if $\mathbf{A}^{-1} \geq 0$

Theorem 2.6 *If a matrix* **A** *is irreducibly diagonally dominant and has nonpositive off-diagonal elements, then* **A** *is monotone*

Theorem 2.7 *If the matrices* **A** *and* **B** *are monotone and* **B** $\leq$ **A**, *then* $\mathbf{B}^{-1} \geq \mathbf{A}^{-1}$

3 RESULTS

3.1 Convergence Analysis

The two-step Obrechkoff method considered in this work is of the form

$$u_{n-1} - 2u_n + u_{n+1} = \frac{1}{252}h^2\left(11f_{n-1} + 230f_n + 11f_{n+1}\right) - \frac{1}{15120}h^4\left(13f^{(2)}_{n-1} - 626f^{(2)}_n + 13f^{(2)}_{n+1}\right) \tag{2}$$

Applying (2) to (1) yields the difference scheme

$$-u_{n-1} + 2u_n - u_{n+1} + \frac{1}{252}h^2\left(11f_{n-1} + 230f_n + 11f_{n+1}\right) - \frac{1}{15120}h^4\left(13f^{(2)}_{n-1} - 626f^{(2)}_n + 13f^{(2)}_{n+1}\right) = 0, \quad n = 1,2,\cdots,N, \tag{3}$$

and the boundary condition becomes

$$u_0 = \eta_1, \quad u_{N+1} = \eta_2 \tag{4}$$

From the above, the exact solution $u(t)$ of (1) satisfies

$$\begin{aligned} &-u(t_{n-1}) + 2u(t_n) - u(t_{n+1}) + \\ &\frac{1}{252}h^2\left(11f\left(t_{n-1}, u\left(t_{n-1}\right)\right) + 230f\left(t_n, u\left(t_n\right)\right) + \right. \\ &\left. 11f\left(t_{n+1}, u\left(t_{n+1}\right)\right)\right) - \\ &\frac{1}{15120}h^4\left(13f^{(2)}\left(t_{n-1}, u\left(t_{n-1}\right)\right) - 626f^{(2)}\left(t_n, u\left(t_n\right)\right) + \right. \\ &\left. 13f^{(2)}\left(t_{n+1}, u\left(t_{n+1}\right)\right)\right) + T_n = 0 \end{aligned} \tag{5}$$

where T_n is the truncation error. Subtracting (5) from (3), applying the Mean Value Theorem and substituting $\varepsilon_n = u_n - u(t_n)$, the error equation is obtained as

$$\begin{aligned} & -\varepsilon_{n-1} + 2\varepsilon_n - \varepsilon_{n+1} + \\ & \frac{1}{252}h^2\left(11\varepsilon_{n-1}f_{u_{n-1}} + 230\varepsilon_n f_{u_n} + 11\varepsilon_{n+1}f_{u_{n+1}}\right) - \\ & \frac{1}{15120}h^4\left(13\varepsilon_{n-1}f^{(2)}_{u_{n-1}} - 626\varepsilon_n f^{(2)}_{u_n} + 13\varepsilon_{n+1}f^{(2)}_{u_{n+1}}\right) - \\ & T_n = 0, \quad n = 1, 2, \cdots, N \end{aligned} \tag{6}$$

where the truncation error is given by

$$\begin{aligned} T &= \frac{59}{76204800}h^{10}u^{10}(\xi), \text{ and} \\ \|\mathbf{T}\| &\leq \frac{59}{76204800}h^{10}\max_{\xi\in[a,b]}\left|u^{(10)}(\xi)\right| \end{aligned} \tag{7}$$

In matrix notation, (6) can be written as

$$\mathbf{ME} = \mathbf{T} \tag{8}$$

where

$$\mathbf{M} = \mathbf{J} + \mathbf{K} + \mathbf{L},$$
$$\mathbf{E} = [\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N]^T,$$
$$\mathbf{T} = [T_1, T_2, \cdots, T_N]^T,$$

$$\mathbf{J} = \begin{pmatrix} 2 & -1 & & & \mathbf{0} \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & & \\ \mathbf{0} & & & -1 & 2 \end{pmatrix} \tag{9}$$

$$\mathbf{K} = \frac{h^2}{252}\begin{pmatrix} 230f_{u_1} & 11f_{u_2} & & & \mathbf{0} \\ 11f_{u_1} & 230f_{u_2} & 11f_{u_3} & & \\ & 11f_{u_2} & 230f_{u_3} & 11f_{u_4} & \\ & & \ddots & & \\ \mathbf{0} & & & 11f_{u_{N-1}} & 230f_{u_N} \end{pmatrix} \tag{10}$$

$$\mathbf{L} = \frac{h^4}{15120}\begin{pmatrix} 626f^{(2)}_{u_1} & -13f^{(2)}_{u_2} & & & \mathbf{0} \\ -13f^{(2)}_{u_1} & 626f^{(2)}_{u_2} & -13f^{(2)}_{u_3} & & \\ & -13f^{(2)}_{u_2} & 626f^{(2)}_{u_3} & -13f^{(2)}_{u_4} & \\ & & \ddots & & \\ \mathbf{0} & & & -13f^{(2)}_{u_{N-1}} & 626f^{(2)}_{u_N} \end{pmatrix} \tag{11}$$

Since $f_{u_n} > 0, \ n = 1, 2, \cdots N$ then $\mathbf{K} + \mathbf{L} \geq \mathbf{0}$ and $\mathbf{M} = \mathbf{J} + \mathbf{K} + \mathbf{L} \geq \mathbf{J}$. Clearly, $\mathbf{J}$ is monotone. Now, the off-diagonal elements of $\mathbf{M}$ are

$$-1 + \frac{11}{252}h^2 f_{u_{n-1}} - \frac{13}{15120}h^4 f^{(2)}_{u_{n-1}}$$

and

$$-1 + \frac{11}{252}h^2 f_{u_{n+1}} - \frac{13}{15120}h^4 f^{(2)}_{u_{n+1}}$$

, and the diagonal elements are

$$2 + \frac{230}{252}h^2 f_{u_n} + \frac{626}{15120}h^4 f^{(2)}_{u_n}$$

. In order to make $\mathbf{M}$ have non-negative off diagonal elements, we need to choose h such that

$$\left.\begin{array}{l} -1 + \frac{11}{252}h^2 f_{u_{n-1}} - \frac{13}{15120}h^4 f^{(2)}_{u_{n-1}} < 0; \text{and} \\ -1 + \frac{11}{252}h^2 f_{u_{n+1}} - \frac{13}{15120}h^4 f^{(2)}_{u_{n+1}} < 0 \end{array}\right\}$$

This means that h be chosen such that

$$-1 + \frac{11}{252}h^2 f_u - \frac{13}{15120}h^4 f^{(2)}_u < 0 \quad \text{over } [a, b]$$

and

$$\begin{aligned} & 2 + \frac{230}{252}h^2 f_{u_n} + \frac{626}{15120}h^4 f^{(2)}_{u_n} \\ \geq & \left|-1 + \frac{11}{252}h^2 f_{u_{n-1}} - \frac{13}{15120}h^4 f^{(2)}_{u_{n-1}}\right| + \\ & \left|-1 + \frac{11}{252}h^2 f_{u_{n+1}} - \frac{13}{15120}h^4 f^{(2)}_{u_{n+1}}\right| \end{aligned} \tag{12}$$

For the above choice of h, $\mathbf{M}$ is irreducibly diagonally dominant and monotone. Since $\mathbf{M} \geq \mathbf{J}$, we have that $\mathbf{0} < \mathbf{M}^{-1} \leq \mathbf{J}^{-1}$. From (8), we have that

$$\mathbf{E} = \mathbf{M}^{-1}\mathbf{T} \tag{13}$$
$$\Rightarrow \|\mathbf{E}\| \leq \|\mathbf{M}^{-1}\|\|\mathbf{T}\| \leq \|\mathbf{J}^{-1}\|\|\mathbf{T}\| \tag{14}$$
$$\leq \left(\frac{(b-a)^2}{8h^2}\right)\left(\frac{59}{76204800}h^{10}\max_{\xi\in[a,b]}\left|u^{(10)}(\xi)\right|\right) \tag{15}$$
$$= \frac{59(b-a)^2}{609638400}h^8\max_{\xi\in[a,b]}\left|u^{(10)}(\xi)\right| \tag{16}$$

It follows that the method is of order eight and

$$\lim_{h\to 0}\|\mathbf{E}\| = 0 \tag{17}$$
$$\Rightarrow \lim_{h\to 0} u_j = u(t_j) \tag{18}$$

This establishes the convergence of (2).

3.2 Stability Analysis

Theorem 3.1 *A method with stability function $R_{mm}(\lambda^2)$ has an interval of periodicity $(0, \lambda_0^2)$ if $\left|R_{mm}(\lambda^2)\right| < 1$ for $0 < \lambda^2 < \lambda_0^2$.*

Theorem 3.2 *A method with stability function $R_{mm}(\lambda^2)$ is P-stable if $\left|R_{mm}(\lambda^2)\right| < 1$ for all real $\lambda \neq 0$.*

Applying (2) to the test problem

$$u'' = -k^2 u, \tag{19}$$

results in

$$\begin{aligned} & u_{n-1} - 2u_n + u_{n+1} \\ & - \frac{1}{252}\lambda^2 \left(11u_{n-1} + 230u_n + 11u_{n+1}\right) - \\ & \frac{1}{15120}\lambda^4 \left(13u_{n-1} - 626u_n + 13u_{n+1}\right) \end{aligned} \quad (20)$$

where $\lambda = kh$. Rearranging and simplifying (20) gives

$$u_{n-1} - 2R_{22}u_n + u_{n+1} \quad = \quad 0,$$

where

$$R_{22}(\lambda^2) \quad = \quad \frac{1 - \frac{115}{252}\lambda^2 + \frac{313}{15120}\lambda^4}{1 + \frac{11}{252}\lambda^2 + \frac{13}{15120}\lambda^4}. \quad (21)$$

The rational expression (21) is the stability function of (2). From the above, it is clear that the (2) is not *P-stable* but has a large interval of periodicity, i.e $[0, 25.2]$

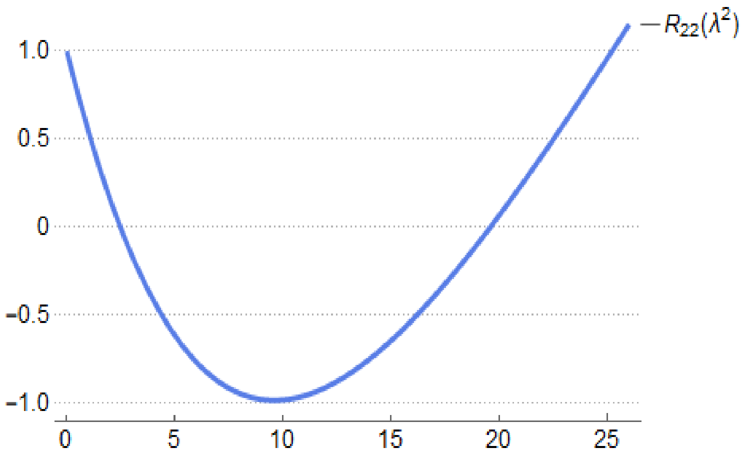

Figure 1: The stability function R_{22} as a function of λ^2

4 CONCLUSION

In this work, we have established the necessary conditions for the two-step Obrechkoff method (2) to be convergent using the properties of matrices. In the stability analysis, we have shown that the method is not *P-stable* but has a large interval of periodicity.

AUTHORS' CONTRIBUTIONS

All authors participated actively in this research work and the writing of the manuscript.

CONSENT (WHERE EVER APPLICABLE)

Consent form has been approved by all authors.

REFERENCES

[1] Lambert, J.D., *Computational Methods in ODEs*, Wiley, New York. 1973.

[2] Stiefel T. E., Bettis D. G., *Stabilization of Cowell's method*, Numer. Math., 1969; 13(4):154–175.

[3] Van Dooren R., *Stabilization of Cowell's classical finite difference method for numerical integration*, J. Comput. Phys., 1974; 16:186–192.

[4] Collatz L., *The Numerical Treatment of Differential Equations*, Springers, 1960.

[5] Numerov, Boris Vasil'evich, *A method of extrapolation of perturbations*, Monthly Notices of the Royal Astronomical Society, 1924; 84:592–601.

[6] Numerov, Boris Vasil'evich, *Note on the numerical integration of* $d^2x/dt^2 = f(x,t)$, Z. Astronomische Nachrichten, 1927; 230:359–364.

[7] Coleman J. P., *Numerical methods for* $y'' = f(x,y)$ *via via rational approximations for the cosine*, IMA J. Numer. Anal. 1989; 9:145–165.

[8] Obrechkoff N., *Sur les quadrature mecanique*, Spisanie Bulgar. Akad. Nauk, 1942; 65:191–289.

[9] Van Daele M., Berghe G. V., *P-stable obrechkoff methods of arbitrary order for second-order differential equations*, Numerical Algorithms, 2007; 44:115–131.

[10] Ash J. H., *Analysis of multistep methods for special second–order ordinary differential equations*, Ph.D Thesis, University of Toronto, Canada. 1969