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ORIGINAL RESEARCH

Construction and Implementation of Optimal 8–Step Linear Multi-step method

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Abstract:
Introduction: Many problems in science and engineering can be formulated as ordinary differential equations. The analytical methods of solving differential equations are applicable only to a selected class of differential equations. Quite often, equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods for solving such differential equations. Linear multistep methods are very popular for solving first order initial value problems.
Aims: In this paper, the optimal 8–step linear multistep method for solving $y' = f(x, y)$ is constructed and implemented.
Materials and Methods: The construction was carried out using the technique based on the Taylor expansion of $y(x + jh)$ and $y'(x + jh)$ about $x + th$, where *t* need not necessarily be an integer.
Results: The consistency, stability and convergence of the proposed method are investigated. To investigate the accuracy of the method, a comparison with the classical 8-stage Runge–Kutta method is carried out on two numerical examples.
Conclusion: In this work, the procedure for the construction of an optimal 8-step linear multistep method for first-order differential equations has been presented. The constructed method is consistent and zero-stable. Hence it is convergent. The accuracy of the method compared with the well-known Runge-Kutta method is demonstrated by its application to two test problems.
Keywords: Linear Multistep Method, Optimal, Consistency, Stability, Convergence, Accuracy

All co-authors agreed to have their names listed as authors.

1 INTRODUCTION

Many problems in science and engineering can be formulated as ordinary differential equations. The analytical methods of solving differential equations are applicable only to a selected class of differential equations. Quite often, equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods for solving such differential equations. Linear multistep methods are very popular for solving first order initial value problems.

Traditionally, they are used to solve higher order ordinary differential equations by first reducing them to a system of first order. This approach has been extensively discussed in [1, 2, 3]. However, the method of reducing to a system of first order has some serious drawback which includes wastage of human effort and computer time [4]. The general *k*-step method or linear multistep method of step number *k* is given as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} \quad = \quad h \sum_{j=0}^{k} \beta_j y'_{n+j}, \tag{1}$$

where α_j and β_j are uniquely determined. The linear multistep method in equation (1) generates schemes which are used to solve first-order ordinary differential equations. Various form of this Linear multistep method has been developed [3, 2, 5, 6]. Other researcher like the author in [6] derived a 6-step linear multistep method of order *8*. The derived scheme was compared with the analytic method. In this study, we shall develop an optimal *8*-step linear multistep method of order *10* for the solution of ordinary differential equation using Taylor series as the basis function.

2 MATERIAL AND METHODS

2.1 Construction of Proposed Method

The form of the linear multistep method to be constructed in this work is

$$\sum_{i=0}^{k=8} \alpha_i y_{n+i} \quad = \quad h \sum_{i=0}^{k=8} \beta_i y'_{n+i}. \tag{2}$$

Associated with (2) is the first characteristics polynomial

$$\rho(\xi) \quad = \quad \sum_{i=0}^{k=8} \alpha_i \xi^i \tag{3}$$

In order to make (2) an optimal method, all the roots of (3) must lie on the unit circle in the complex plane. With $\xi = 1$ as a root of $\rho(\xi) = 0$, we have that

$$\sum_{i=0}^{k=8} \alpha_i \quad = \quad 0, \tag{4}$$

which established the consistency of (2). Since (3) is a polynomial of degree 8, it must have another real root on the unit circle. In order for (2) to be zero–stable, the this other real root must be -1 and the remaining 6 roots must be complex.

Thus we have

$$\left.\begin{array}{llll} \xi_1 = +1, & \xi_2 = -1, & \xi_3 = e^{i\theta_1}, & \xi_4 = e^{-i\theta_1}, \\ \xi_5 = e^{i\theta_2}, & \xi_6 = e^{-i\theta_2}, & \xi_7 = e^{i\theta_3}, & \xi_8 = e^{-i\theta_3} \end{array}\right\}, \qquad 0 < \theta_1, \theta_2, \theta_3 < \pi \tag{5}$$

Therefore,

$$\begin{aligned} \rho(\xi) = (\xi - 1)(\xi + 1)\left(\xi - e^{i\theta_1}\right)\left(\xi - e^{-i\theta_1}\right) \times \\ \left(\xi - e^{i\theta_2}\right)\left(\xi - e^{-i\theta_2}\right)\left(\xi - e^{i\theta_3}\right)\left(\xi - e^{-i\theta_3}\right) \end{aligned} \tag{6}$$

which upon simplification results in

$$\begin{aligned} &\rho(\xi) = \xi^8 + \xi^7 \left(-2\cos(\theta_1) - 2\cos(\theta_2) - 2\cos(\theta_3)\right) + \\ &\xi^6 \left(4\cos(\theta_1)\cos(\theta_2) + 4\cos(\theta_3)\cos(\theta_2) + 4\cos(\theta_1)\cos(\theta_3) + 2\right) + \\ &\xi^5 \left(-8\cos(\theta_2)\cos(\theta_3)\cos(\theta_1) - 2\cos(\theta_1) - 2\cos(\theta_2) - 2\cos(\theta_3)\right) + \\ &\xi^3 \left(8\cos(\theta_2)\cos(\theta_3)\cos(\theta_1) + 2\cos(\theta_1) + 2\cos(\theta_2) + 2\cos(\theta_3)\right) + \\ &\xi^2 \left(-4\cos(\theta_1)\cos(\theta_2) - 4\cos(\theta_3)\cos(\theta_2) - 4\cos(\theta_1)\cos(\theta_3) - 2\right) + \\ &\xi \left(2\cos(\theta_1) + 2\cos(\theta_2) + 2\cos(\theta_3)\right) - 1. \end{aligned} \tag{7}$$

Setting $\cos(\theta_1) = \lambda_1$, $\cos(\theta_2) = \lambda_2$ and $\cos(\theta_3) = \lambda_3$ in (7) and comparing the coefficients of powers of ξ in the resulting expression with (3), we obtain

$$\left.\begin{array}{l} \alpha_8 = +1, \\ \alpha_7 = -2(\lambda_1 + \lambda_2 + \lambda_3), \\ \alpha_6 = 4\lambda_2\lambda_3 + 4\lambda_1(\lambda_2 + \lambda_3) + 2, \\ \alpha_5 = -2(\lambda_2 + \lambda_3 + \lambda_1(4\lambda_2\lambda_3 + 1)), \\ \alpha_4 = 0 \\ \alpha_3 = 2(\lambda_2 + \lambda_3 + \lambda_1(4\lambda_2\lambda_3 + 1)), \\ \alpha_2 = -2(2\lambda_2\lambda_3 + 2\lambda_1(\lambda_2 + \lambda_3) + 1), \\ \alpha_1 = 2(\lambda_1 + \lambda_2 + \lambda_3), \\ \alpha_0 = -1 \end{array}\right\} \tag{8}$$

Since the stepnumber = *k* = *8* is even, we now require that the proposed optimal method have order *k* + *2* = *10*. Thus, the order requirement (see [2]) with *t=4* in terms of the coefficients gives

$$
\begin{aligned}
&\beta_0+\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+\beta_6+\beta_7+\beta_8=\\
&-16(\lambda_1-1)(\lambda_2-1)(\lambda_3-1)\\
&4\beta_0+3\beta_1+2\beta_2+\beta_3=\beta_5+2\beta_6+3\beta_7+4\beta_8\\
&3(16\beta_0+9\beta_1+4\beta_2+\beta_3+\beta_5+4\beta_6+9\beta_7+16\beta_8)=\\
&-16(7\lambda_3+\lambda_2(7-4\lambda_3)+\lambda_1(\lambda_2(\lambda_3-\\
&4)-4\lambda_3+7)-10)+64\beta_0+27\beta_1+8\beta_2+\beta_3=\\
&\beta_5+8\beta_6+27\beta_7+64\beta_8\\
&1280\beta_0+405\beta_1+80\beta_2+5\beta_3+5\beta_5+80\beta_6+\\
&405\beta_7+1280\beta_8=16(-61\lambda_3+\lambda_2(16\lambda_3-61)+\\
&\lambda_1(-\lambda_2(\lambda_3-16)+16\lambda_3-61)+136)\\
&1024\beta_0+243\beta_1+32\beta_2+\beta_3=\beta_5+32\beta_6+243\beta_7+1024\beta_8\\
&28672\beta_0+5103\beta_1+448\beta_2+7\beta_3+7\beta_5+448\beta_6+\\
&5103\beta_7+28672\beta_8=(-547\lambda_3+\lambda_2(64\lambda_3-547)+\\
&\lambda_1(-\lambda_2(\lambda_3-64)+64\lambda_3-547)+2080)\\
&16384\beta_0+2187\beta_1+128\beta_2+\beta_3=\\
&\beta_5+128\beta_6+2187\beta_7+16384\beta_8\\
&589824\beta_0+59049\beta_1+2304\beta_2+9\beta_3\\
&+9\beta_5+2304\beta_6+59049\beta_7+589824\beta_8=\\
&16(-4921\lambda_3+\lambda_2(256\lambda_3-4921)\\
&+\ \lambda_1(-\lambda_2(\lambda_3-256)+256\lambda_3-4921)+32896)\\
&262144\beta_0+19683\beta_1+512\beta_2+\beta_3=\\
&\beta_5+512\beta_6+19683\beta_7+262144\beta_8
\end{aligned}
\tag{9}
$$

Solving the above system of equation, we get

$$
\begin{aligned}
\beta_0&=\frac{1}{14175}(\lambda_1(188+52\lambda_3+\lambda_2(52+23\lambda_3))+\\
&\quad 2(1991+94\lambda_3+\lambda_2(94+26\lambda_3)))\\
\beta_1&=\frac{-1}{14175}(2(-11552+4922\lambda_3+\lambda_2(4922+448\lambda_3)+\\
&\quad \lambda_1(4922+448\lambda_3+\lambda_2(448+167\lambda_3))))\\
\beta_2&=\frac{1}{14175}(4(1819-9484\lambda_3+\lambda_2(-9484+5494\lambda_3)+\\
&\quad \lambda_1(-9484+5494\lambda_3+\lambda_2(5494+701\lambda_3))))\\
\beta_3&=\frac{1}{14175}(\lambda_1(-27268+\lambda_2(70528-46378\lambda_3)+70528\lambda_3)+\\
&\quad 4(19312-6817\lambda_3+\lambda_2(-6817+17632\lambda_3)))\\
\beta_4&=\frac{-1}{2835}(2(-358+\lambda_2(7708-4348\lambda_3)+7708\lambda_3+\\
&\quad \lambda_1(7708-4348\lambda_3+\lambda_2(-4348+13903\lambda_3))))
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\beta_5&=\frac{1}{14175}((-27268+\lambda_2(70528-46378\lambda_3)+70528\lambda_3)+\\
&\quad \lambda_1 4(19312-6817\lambda_3+\lambda_2(-6817+17632\lambda_3)))\\
\beta_6&=\frac{1}{14175}(4(1819-9484\lambda_3+\lambda_2(-9484+5494\lambda_3)+\\
&\quad \lambda_1(-9484+5494\lambda_3+\lambda_2(5494+701\lambda_3))))\\
\beta_7&=\frac{-1}{14175}(2(-11552+4922\lambda_3+\lambda_2(4922+448\lambda_3)+\\
&\quad \lambda_1(4922+448\lambda_3+\lambda_2(448+167\lambda_3))))\\
\beta_8&=\frac{1}{14175}(\lambda_1(188+52\lambda_3+\lambda_2(52+23\lambda_3))+\\
&\quad 2(1991+94\lambda_3+\lambda_2(94+26\lambda_3)))
\end{aligned}
\tag{11}
$$

Following [2], the expression for the error constant (lte) associated with this method is given by

$$
\begin{aligned}
lte&=\frac{1}{935550}(-4(305\lambda_3+\lambda_2(124\lambda_3+305)+1246)\\
&\quad -\lambda_1(496\lambda_3+\lambda_2(263\lambda_3+496)+1220))
\end{aligned}
\tag{12}
$$

The choice of values of λ_1, λ_2, and λ_3 is critical to the zerostability of (2). To ensure zero-stability of the method, we choose the following values: $\lambda_1=\frac{1}{\sqrt{2}}$, $\lambda_2=0$, $\lambda_3=-\frac{1}{\sqrt{2}}$. The above choice makes (2) to be zerostable. Substituting the values of λ_1, λ_2, and λ_3 into $\alpha_i' s$ and $\beta_i' s$, we get the resulting scheme

$$
\begin{aligned}
&y_{n+8}-y_n=\frac{3956}{14175}f_n+\frac{23552}{14175}f_{n+1}-\frac{3712}{14175}f_{n+2}+\\
&\frac{41984}{14175}f_{n+3}-\frac{3632}{2835}f_{n+4}+\frac{41984}{14175}f_{n+5}-\\
&\frac{3712}{14175}f_{n+6}+\frac{23552}{14175}f_{n+7}+\frac{3956}{14175}f_{n+8}
\end{aligned}
\tag{13}
$$

Since the order *p=10* of the method is greater than one and the method already is zerostable, then the method (13) is convergent. We shall refer to the new method as (*Opt8sM*).

3 RESULTS

3.1 Computational Analysis

The aim of the computational analysis carried out in this section is to investigate the accuracy and efficiency of the proposed method compared with some existing methods. The proposed scheme is implemented on two problems that have been studied in the literature [7]. Since our proposed method is of an 8-step method, then it becomes logical to compare it with method of equivalent stage hence, the choice of explicit 8-stage Runge-Kutta *(E8RK)* method.

3.1.1 Problem 1

Consider the linear problem

$$
y'(x) \;=\; x+y, \quad y(0)=1 \qquad x\in[0,1]
$$

with exact solution

$$y(x) = 2\,\text{exp}(x) - x - 1;$$

3.2 Problem 2

The second problem considered in this work is given as

$$y'(x) = x^2 y, \quad y(0) = 1 \qquad x \in [0, 1]$$

with exact solution

$$y(x) = \text{exp}\left(\frac{1}{3}x^3\right)$$

4 CONCLUSION

In this work, the procedure for the construction of an optimal 8-step linear multistep method for first-order differential equations has been presented. The constructed method is consistent and zero-stable. Hence it is convergent. The accuracy of the method compared with the well-known Runge-Kutta method is demonstrated by its application to two test problems.

AUTHORS' CONTRIBUTIONS

All authors participated actively in this research work and the writing of the manuscript.

CONSENT (WHERE EVER APPLICABLE)

Consent form has been approved by all authors.

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Table 1: The absolute error of the proposed *Opt8sM* method compared with *(E8RK)* with steplength h=0.1

t_n	$Y_{exact}(t_n)$	$E8RK(t_n)$	$Opt8sM(t_n)$	Abs. Error $E8RK(t_n)$	Abs. Error $Opt8sM(t_n)$
0.8	2.6510819	2.6510819	2.6510819	7.2495787E-11	2.1316726E-11
0.9	3.0192062	3.0192062	3.0192062	9.0135899E-11	2.4826807E-11
1.	3.4365637	3.4365637	3.4365637	1.1068435E-10	3.8390624E-11

Table 2: The absolute error of the proposed *Opt8sM* method compared with *(E8RK)* with steplength h=0.0625

t_n	$Y_{exact}(t_n)$	$E8RK(t_n)$	$Opt8sM(t_n)$	Abs. Error $E8RK(t_n)$	Abs. Error $Opt8sM(t_n)$
0.5	1.7974425	1.7974425	1.7974425	2.1573854E-12	4.3032244E-13
0.5625	1.9476093	1.9476093	1.9476093	2.5839331E-12	5.6310512E-13
0.625	2.1114919	2.1114919	2.1114919	3.0562219E-12	9.1393559E-13
0.6875	2.2899749	2.2899749	2.2899749	3.5784709E-12	9.7699626E-13
0.75	2.484	2.484	2.484	4.1553427E-12	1.458389E-12
0.8125	2.6945696	2.6945696	2.6945696	4.7921667E-12	1.6253665E-12
0.875	2.9227506	2.9227506	2.9227506	5.4933835E-12	2.0223823E-12
0.9375	3.1696789	3.1696789	3.1696789	6.2660988E-12	2.4273916E-12
1.	3.4365637	3.4365637	3.4365637	7.1151973E-12	1.085354E-12

Table 3: The absolute error of the proposed *Opt8sM* method compared with *(E8RK)* with steplength h=0.05

t_n	$Y_{exact}(t_n)$	$E8RK(t_n)$	$Opt8sM(t_n)$	Abs. Error $E8RK(t_n)$	Abs. Error $Opt8sM(t_n)$
0.40	1.021563	1.021563	1.021563	2.384759E-13	1.317613E-12
0.45	1.030841	1.030841	1.030841	3.397283E-13	1.762591E-12
0.50	1.042547	1.042547	1.042547	4.691803E-13	2.304823E-12
0.55	1.057025	1.057025	1.057025	6.317169E-13	3.035128E-12
0.60	1.074655	1.074655	1.074655	8.333334E-13	3.981260E-12
0.65	1.095862	1.095862	1.095862	1.080913E-12	5.300427E-12
0.70	1.121126	1.121126	1.121126	1.382228E-12	7.093659E-12
0.75	1.150993	1.150993	1.150993	1.746159E-12	9.588108E-12
0.80	1.186095	1.186095	1.186095	2.182032E-12	1.414402E-11
0.85	1.227167	1.227167	1.227167	2.700507E-12	1.939360E-11
0.90	1.275069	1.275069	1.275069	3.312906E-12	2.664402E-11
0.95	1.330815	1.330815	1.330815	4.031664E-12	3.697487E-11
1.00	1.395612	1.395612	1.395612	4.870104E-12	5.145950E-11