Research Article
Journal of Research and Review in Science, 149–155 149–155

Volume 4, December 2017
DOI:10.36108/jrrslasu/7102/40(0112)

ORIGINAL RESEARCH

Existence of Periodic Properties of Solutions of Certain Autonomous Third Order Nonlinear Differential Equations

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Funding information
This research work is self sponsored.

Abstract:
Introduction: It is well known that the periodic properties of solutions play a key role in characterizing the behavior of solutions of nonlinear differential equations. With reference to our observation in the relevant literature, work on the periodic properties of solutions for certain autonomous third-order nonlinear differential equations are very scarce.
Aims: In this work, we establish sufficient conditions that ensure the existence of periodic (or almost periodic) solutions of this class of differential equations.
Materials and Methods: The Lyapunov's second or direct method, a complete Lyapunov function was constructed and used to obtain our results.
Results: Sufficient Conditions were obtained for the existence of periodic and almost periodic solutions for certain autonomous third-order nonlinear differential equation.
Conclusion: The results extend and improve on some earlier results in the literature.
Keywords: Periodic solutions, Almost periodic solutions; Third order nonlinear differential equations, Lyapunov's method.

All co-authors agreed to have their names listed as authors.

1 INTRODUCTION

This paper considers the periodic or almost periodic solutions of the autonomous third-order differential equation

$$\dddot{x} + \varphi(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t,x,\dot{x},\ddot{x}), \qquad (1)$$

in which φ, g, h and p depend on the arguments displayed explicitly and dots denote differentiation with respect to t. Moreover, the existence and the uniqueness of solutions of (1) will be assumed.

Equations of the form (1) do arise in some aspect of applied sciences such as after effect, nonlinear oscillations, biological systems and equations with deviating arguments (see [1], [2] and [3]) and an effective method for studying the qualitative properties of solutions of such nonlinear equations is still the Lyapunov's direct method (see [4], [5], [6], [7], [8], [9], [10], [11], [12]). Many of these results on stability, boundedness, convergence of solutions exist for more general or special cases of (1) and are summarized in [13]. The search for periodic solutions and the examination of their behavior is of interest because of the mathematical description of nonlinear systems and the determination of periodic regime of real physical systems modeled into nonlinear differential equations. The periodic properties of solutions for some kind of nonlinear third-order scalar differential equations has been addressed by only a few, for example Andres [14], Ezeilo [15], Ezeilo [16], Ezeilo and Nkashama [17], Ogbu [18], Pliss [19] and Villari [20]. However, the methods employed by [14, 15, 16, 17, 18, 19] and [20] were based on fixed point theorems, the Brouwer fixed-point theorem and the Lerray-Schauder fixed point theorem also referred to as the "non-Routh Hurwitz" direction in proving the existence of periodic solutions of third-order differential equations.

In this work, we consider a somewhat different approach to "non-Routh Hurwitz" direction in establishing the existence of periodic or almost periodic solutions of equation (1) if p is periodic or almost periodic due to the presence of the perturbation r. To the best of our knowledge, results in the direction of Routh-Hurwitz do not exist ! The problem, however, in using the Lyapunov method approach to establish the existence of periodic solutions is the difficulty in constructing a suitable complete Lyapunov function (see [4]). Our results will be in the direction of Routh Hurwitz and may be applied to the spatial discretion of some third-order differential equations (see [21]).

2 MATERIAL AND METHODS

2.1 Definitions

Definition 2.1 *A continuous function $f:\mathbb{R}\to x$ is called almost periodic if for each $\varepsilon>0$ there exists $\ell(\varepsilon)>0$ such that every interval of length $\ell(\varepsilon)$ contains a number τ with property that*

$$|f(t+\tau)-f(t)|<\varepsilon \ \ for\ each\ t\in\mathbb{R}.$$

Definition 2.2 *A continuous function $f:\mathbb{R}\to x$ is said to be periodic with period ω for all $t\in\mathbb{R}$ such that*

$$f(t+\omega)=f(t) \ \ for\ all\ t\in\mathbb{R}.$$

Assume now that r is the perturbation such that p is continuous function $p(t,x,\dot{x},\ddot{x})$ is separable in the form

$$p(t,x,\dot{x},\ddot{x}) = q(t) + r(t,x,\dot{x},\ddot{x}),$$

with $q(t)+r(t,0,0,0)$ continuous in their respective arguments, where

$$|q(t)| = \int_0^t |q(s)| \le D_1, \ \ D_1>0.$$

Our main result is the following

2.2 Main result

Theorem 2.3 *Further to the basic assumptions imposed on the functions φ, g, h and p in equation (1). We also assume that a, b, c, δ_o are positive constants and $h(0)=0$, then the following conditions hold:*

(i) $\varphi(x,y)>a$, $\frac{g(y)}{y}\ge b$, $h_x(x)\le c$ *and* $ab-c>0$ *for all* x,y,z;

(ii) $\frac{h(x)}{x}\ge\delta_o$, *for all* $x\neq 0$ *;*

(iii) $y\varphi_x(x,y)\le 0$, *for all x,y;*

(iv) $p(t,x,y,z)\equiv q(t)+r(t,x,y,z)$ *satisfies*

$$|r(t,x_1,y_1,z_1+q)-r(t,x_2,y_2,z_2+q)| \le \phi\{|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\}.$$

for arbitrary t and $x_1,x_2,y_1,y_2,z_1,z_2\in\mathbb{R}$, with $\phi(t)$ a continuous function satisfying

$$\int_{-\infty}^{\infty}\phi^{\gamma}(t)dt<\infty,$$

for some constant γ in the range $1\le\gamma\le 2$. Suppose further that there exists a solution $x(t)$ of equation (1) such that

$$|x(t)|^2+|\dot{x}(t)|^2+|\ddot{x}(t)|^2\le D_2.$$

Then,

1. *If $q(t)$ is almost periodic and $r(t,x,\dot{x},\ddot{x})$ is almost periodic in t, for $|x(t)|^2+|\dot{x}(t)|^2+|\ddot{x}(t)|^2\le D_2$, then $x(t)$ is almost periodic in t.*

2. *If $q(t)$ and $r(t, x, \dot{x}, \ddot{x})$ are periodic in t, with period ω, for $|x(t)|^2 + |\dot{x}(t)|^2 + |\ddot{x}(t)|^2 \le D_2$, then $x(t)$ is periodic with period ω.*

Now, let equation (1) be replaced with the equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z + q, \\ \dot{z} &= -\varphi(x,y)z - g(y) - h(x) + p(t,x,y,z+q) \\ &- \varphi(x,y)q. \end{aligned} \tag{2}$$

2.3 Preliminary results

Let (x, y, z) be any solution of system (2). Our main tool is the following scalar function, defined by

$$V = V_1 + V_2, \tag{3}$$

where V_1 and V_2 are given by

$$\begin{aligned} 2V_1 &= 2\int_0^x h(\xi)d\xi + 2\int_0^y \varphi(x,\sigma)d\sigma + 2\alpha\int_0^y g(\sigma)d\sigma \\ &+ \alpha z^2 + 2yz + 2\alpha y h(x) \end{aligned}$$

and

$$\begin{aligned} 2V_2 &= \beta\ell b x^2 + 2a\int_0^x h(\xi)d\xi + 2a\int_0^y \varphi(x,\sigma)d\sigma \\ &+ 2\alpha\int_0^y g(\sigma)d\sigma + z^2 + 2\beta\ell xy + 2ayz \\ &+ 2yh(x) + 2\beta\ell xz, \end{aligned}$$

$\alpha > 0$ is a fixed constant chosen such that

$$\frac{1}{a} < \alpha < \frac{b}{c} \tag{4}$$

and $0 < \beta < 1$ chosen such that

$$\begin{aligned} \beta &< \min\left\{ \frac{ab - c}{\ell\left[a + \delta_o^{-1}\left(\frac{g(y)}{y} - b\right)^2\right]}, \frac{1}{a}, \right. \\ &\quad \left. \frac{a\alpha - 1}{ab\alpha}, \frac{\delta_o(a\alpha - 1)}{\ell[\varphi(x,y) - a]^2} \right\}. \end{aligned} \tag{5}$$

The function V can be re-arranged as follows:

$$\begin{aligned} 2V_1 &= \left\{2\int_0^x h(\xi)d\xi - \frac{\alpha}{b}h^2(x)\right\} + \alpha b\left\{y + \frac{h(x)}{b}\right\}^2 \\ &+ \{2\int_0^y \varphi(x,\sigma)d\sigma - \alpha^{-1}y^2\} + \alpha\{z + \alpha^{-1}y\}^2 \\ &+ \alpha\{\int_0^y g(\sigma)d\sigma - by^2\} \end{aligned}$$

and

$$\begin{aligned} 2V_2 &= \beta\ell(b - \beta\ell)x^2 + a\{2\int_0^x h(\xi)d\xi - \ell^{-1}h^2(x)\} \\ &+ \ell\{a^{\frac{1}{2}}y + \ell^{-1}a^{-\frac{1}{2}}h(x)\}^2 + \{2\int_0^y g(\sigma)d\sigma - \ell a^{-1}y^2\} \\ &+ a\{\int_0^y \varphi(x,\sigma)d\sigma - ay^2\} + \{\beta\ell x + ay + z\}^2. \end{aligned}$$

The term

$$2\int_0^x h(\xi)d\xi - \frac{\alpha}{b}h^2(x),$$

in the re-arrangement of $2V_1$ becomes

$$2\int_0^x (1 - \frac{\alpha}{b}h_\xi(\xi))h(\xi)d\xi - \frac{\alpha}{b}h^2(0),$$

using the hypotheses (i) and (ii) of Theorem 1, we get

$$2\int_0^x h(\xi)d\xi - \frac{\alpha}{b}h^2(x) \ge \left(1 - \frac{\alpha}{b}c\right)\delta_o x^2.$$

Similarly, the term

$$2\int_0^x h(\xi)d\xi - \ell^{-1}h^2(x),$$

in the re-arrangement of $2V_2$ becomes

$$2\int_0^x h(\xi)d\xi - \ell^{-1}h^2(x) \ge \left(1 - \frac{c}{\ell}\right)\delta_o x^2.$$

Combining these estimates, we obtain for

$$\begin{aligned} V &\ge \left\{\left(1 - \frac{\alpha}{b}c\right)\delta_o + \beta\ell(b - \beta\ell) + \left(1 - \frac{c}{\ell}\right)\delta_o\right\}x^2 \\ &+ \left\{\left(a - \frac{1}{\alpha} - \beta\ell\right) + \left(b - \frac{\ell}{a}\right)\right\}y^2 \\ &+ \alpha\left(z + \frac{1}{\alpha}y\right)^2 + (\beta\ell x + ay + z)^2, \end{aligned}$$

where α, β satisfy (4) and (5) if we chose $\ell = ab$.
Thus, there exists a constant δ_1 small enough such that

$$V \ge \delta_1(x^2 + y^2 + z^2),$$

where

$$\begin{aligned} \delta_1 &= \min\left\{(1 - \frac{\alpha}{b}c)\delta_o + \beta\ell(b - \beta\ell) + (1 - \frac{c}{\ell})\delta_o; (a - \frac{1}{\alpha} - \beta\ell)\right. \\ &+ \left.(b - \frac{\ell}{a}); (\alpha + 1)\right\}. \end{aligned}$$

Also, from (3) and by Schwartz's inequality, we have

$$V \le \delta_2(x^2 + y^2 + z^2),$$

where

$$\delta_2 = \max\Big\{1+a+\delta_o(1+\alpha)+\beta(2+b); 2+a(a+2)+\alpha(b+1)+\beta\ell+b; \alpha+2+a+\beta\ell\Big\}.$$

Hence, V is positive definite and satisfies

$$\delta_1(x^2+y^2+z^2) \leq V \leq \delta_2(x^2+y^2+z^2). \qquad (6)$$

Furthermore, along any solution $(x(t), y(t), z(t))$ of (2), we have

$$\frac{d}{dt}V(x,y,z) = \dot{V}$$

given by

$$\begin{aligned}\dot{V} &= (1+a)y\int_0^y \varphi_x(x,\sigma)d\sigma + (1+\alpha)h_x(x)y^2\\ &- (1+a)\frac{g(y)}{y}y^2\\ &- [\varphi(x,y)-a]z^2 - \alpha\varphi(x,y)z^2 - \beta\ell\Big\{\frac{g(y)}{y}-b\Big\}xy + z^2\\ &- \beta\ell\frac{h(x)}{x}x^2 - \beta\ell[\varphi(x,y)-a]xz + a\beta\ell y^2\\ &+ (1+\alpha)\frac{h(x)}{x}xq(t) + \Big\{(1+\alpha)\frac{g(y)}{y}y - \beta\ell y\Big\}q(t)\\ &+ (1-\alpha a)zq(t)\\ &+ [\beta\ell x + (1+a)y + (1+\alpha)z]r(t,x,y,z+q).\end{aligned}$$

By the hypotheses of the Theorem, we have

$$\begin{aligned}\dot{V} &\leq (1+\alpha)cy^2 - (1+a)by^2 - \alpha az^2 + z^2\\ &- \beta\ell\Big\{\frac{g(y)}{y}-b\Big\}xy\\ &- \beta\ell[\varphi(x,y)-a]xz + \beta\ell\delta_o x^2 + a\beta\ell y^2 - az^2\\ &+ [(1+\alpha)\delta_o x + (b(1+\alpha)-\beta\ell)y + (1-\alpha a)z]q(t)\\ &+ [\beta\ell x + (1+a)y + (1+\alpha)z]r(t,x,y,z+q).\end{aligned}$$

It is obvious that

$$\begin{aligned}\dot{V} = &- \frac{1}{2}\beta\ell\delta_o x^2 - \Big[ab - c - \beta\ell\Big(a + \delta_o^{-1}\Big(\frac{g(y)}{y}-b\Big)\Big)^2\Big]y^2\\ &- (b-\alpha c)y^2 - [a\alpha - 1 - \beta\ell\delta_o^{-1}(\varphi(x,y)-a)^2]z^2 - az^2\\ &- \frac{1}{4}\beta\ell\delta_o\Big[x + 2\delta_o^{-1}\Big(\frac{g(y)}{y}-b\Big)y\Big]^2\\ &- \frac{1}{4}\beta\ell\delta_o[x + 2v^{-1}(\varphi(x,y)-a)z]^2\\ &+ [(1+\alpha)\delta_o x + (b(1+\alpha)-\beta\ell)y + (1-\alpha a)z]q(t)\\ &+ [\beta\ell x + (1+a)y + (1+\alpha)z]r(t,x,y,z+q).\end{aligned}$$

If β satisfies (5), we have that

$$\begin{aligned}\dot{V} \leq &- \frac{1}{2}\beta\ell\delta_o x^2 - (b-\alpha c)y^2 - az^2\\ &+ [(1+\alpha)\delta_o x + (b(1+\alpha)-\beta\ell)y + (1-\alpha a)z]q(t)\\ &+ [\beta\ell x + (1+a)y + (1+\alpha)z]r(t,x,y,z+q).\end{aligned}$$

It follows that

$$\begin{aligned}\dot{V} \leq &- \delta_3(x^2+y^2+z^2) + \delta_4(x^2+y^2+z^2)^{\frac{1}{2}}\\ &+ \delta_5(x^2+y^2+z^2)^{\frac{1}{2}}|r(t,x,y,z+q)|,\end{aligned}$$

where

$$\delta_3 = \min\frac{1}{2}\{\beta\ell\delta_o, 2(b-\alpha c), 2a\},$$

$$\delta_4 = \max\sqrt{3}D_1\{\delta_o(1+\alpha), b(1+\alpha)-\beta\ell, 1-\alpha a\},$$

and

$$\delta_4 = \max\sqrt{3}\{\beta\ell, 1+a, 1+\alpha\}.$$

Thus,

$$\begin{aligned}\dot{V} \leq &- \delta_3(x^2+y^2+z^2)\\ &+ \delta_6(x^2+y^2+z^2)^{\frac{1}{2}}[r(t,x,y,z+q)+1],\end{aligned}$$

where

$$\delta_6 = \max\{\delta_4, \delta_5\}.$$

So that since

$$|r(t,x,y,z+q)| \leq \delta_6\phi(t)[(x^2+y^2+z^2)^{\frac{1}{2}}+1],$$

$$\begin{aligned}\dot{V} \leq &- \delta_3(x^2+y^2+z^2) + \delta_6\phi(t)(x^2+y^2+z^2)\\ &+ \delta_6\phi(t)(x^2+y^2+z^2)^{\frac{1}{2}}.\end{aligned}$$

By (iv) of Theorem 1 and (6), we have

$$\dot{V} \leq - (\delta_7 - \delta_8\phi(t))V + \delta_9 V^{\frac{1}{2}},$$

where $\delta_7 = \frac{\delta_3}{\delta_2}$, $\delta_8 = \frac{\delta_5}{\delta_1}$ $\delta_9 = \frac{\delta_4}{\delta_1}$.

Following the argument used in [5] it can be further verified that

$$\dot{V} \leq - \delta_{10}(x^2+y^2+z^2) + \delta_{11}\phi(t)(x^2+y^2+z^2)^{\frac{1}{2}}|\theta|, \qquad (7)$$

where $\theta = r(t,x_1,y_1,z_1+q) - r(t,x_2,y_2,z_2+q)$ and δ_{10}, δ_{11} are finite constants.

3 RESULTS

3.1 Proof of Theorem 1

Consider the function

$$U(t) = V\big(x(t-\tau)-x(t), y(t-\tau)-y(t), z(t-\tau)-z(t)\big) \qquad (8)$$

where V is the function defined in (3) with x, y, z replaced by $(x(t+\tau)-x(t)), (y(t+\tau)-y(t))$ and $(z(t+\tau)-z(t))$ respectively. Then, by (6) we have positive constants D_3 and D_4 such that

$$D_3 S(t) \leq U(t) \leq D_4 S(t), \tag{9}$$

where

$$S(t) = \{|x(t+\tau)-x(t)|^2 + |y(t+\tau)-y(t)|^2 + |z(t+\tau)-z(t)|^2\}.$$

Differentiating $U(t)$ along the system (2), we get as in (7),

$$\begin{aligned}
\dot{U}(t) &\leq -\delta_{10}\{|x(t+\tau)-x(t)|^2 + |y(t+\tau)-y(t)|^2 \\
&+ |z(t+\tau)-z(t)|^2\} \\
&+ \delta_{11}\{|x(t+\tau)-x(t)|^2 + |y(t+\tau)-y(t)|^2 \\
&+ |z(t+\tau)-z(t)|^2\}^{\frac{1}{2}}|\theta|,
\end{aligned} \tag{10}$$

where $\theta = r((t+\tau), x(t), y(t), z(t) + q(t+\tau) - r(t, x, y, z+q)$ with δ_{10} and δ_{11} being finite constants.
Inequality (10) can be arranged as

$$\begin{aligned}
\dot{U}(t) &\leq -\delta_{10}\{|x(t+\tau)-x(t)|^2 + |y(t+\tau)-y(t)|^2 \\
&+ |z(t+\tau)-z(t)|^2\} \\
&+ \delta_{12}\{|x(t+\tau)-x(t)|^2 + |y(t+\tau)-y(t)|^2 \\
&+ |z(t+\tau)-z(t)|^2\}^{\frac{1}{2}}|\theta| \\
&+ \delta_{13}\{|x(t+\tau)-x(t)|^2 + |y(t+\tau)-y(t)|^2 \\
&+ |z(t+\tau)-z(t)|^2\}^{\frac{1}{2}} \\
&\times |r(t+\tau), x(t), y(t), z(t) + q(t+\tau) - r(t, x(t), y(t), z+q)|
\end{aligned} \tag{11}$$

Since the perturbation r is uniformly almost periodic in t. Then, given arbitrary $\varepsilon > 0$, we can find $\tau > 0$ such that $|q(t+\tau) - q(t)| \leq \ell\varepsilon^2$,

$$|r(t+\tau), x(t), y(t), z(t) + q(t+\tau) - r(t, x(t), y(t), z+q)| \leq \ell\varepsilon^2 \tag{12}$$

where ℓ is a constant whose value will be determined later. Thus, (11) becomes

$$\dot{U}(t) \leq -\delta_{10}S(t) + \delta_{12}S^{\frac{1}{2}}|\theta| + \delta_{13}S^{\frac{1}{2}}(t)\ell\varepsilon^2 \tag{13}$$

By (iv) of Theorem 1,

$$\{|x(t+\tau)-x(t)|^2 + |y(t+\tau)-y(t)|^2 + |z(t+\tau)-z(t)|^2\}^{\frac{1}{2}} \leq \mathcal{D}_2$$

(13) becomes,

$$\dot{U}(t) + \delta_{10}S(t) \leq \delta_{12}^{\frac{1}{2}}|\theta| + \delta_{13}\mathcal{D}_2\ell\varepsilon^2. \tag{14}$$

Let γ be any constant such that $1 \leq \gamma \leq 2$ and set $m = 1 - \frac{1}{2}\gamma$, so that $0 \leq m \leq \frac{1}{2}$.
Then, (14) becomes

$$\dot{U} + \delta_{10}S(t) \leq \delta_{12}S^m U^* + \delta_{13}\mathcal{D}_2\ell\varepsilon^2 \tag{15}$$

and $U^* = S^{(\frac{1}{2}-m)}(|\theta| - \delta_{10}\delta_{12}^{-1}S^{\frac{1}{2}}(t))$.

We consider two cases

(i) $|\theta| \leq \delta_{10}\delta_{12}^{-1}S^{\frac{1}{2}}$ and

(ii) $|\theta| > \delta_{10}\delta_{12}^{-1}S^{\frac{1}{2}}$

separately, we find that in either case, there exists some constants $\delta_{14} > 0$ such that $U^* \leq \delta_{14}|\theta|^{2(1-m)}$. Thus, (15) becomes

$$\frac{dU}{dt} + \delta_{10}S \leq \delta_{15}S^m\phi^{2(1-m)}S^{(1-m)}U(t) + \delta_{13}\mathcal{D}_2\ell\varepsilon^2$$

where $\delta_{15} \geq 2\delta_{12}\delta_{14}$. Using (9) on U, we obtain

$$\frac{dU}{dt} + \big((\delta_{16} - \delta_{17})\phi^{\gamma}(t)\big)U(t) \leq \delta_{13}\mathcal{D}_2\ell\varepsilon^2 \tag{16}$$

where δ_{16}, δ_{17} as positive constants.
On integrating (16) from t_o to t $(t \geq t_o)$, we obtain

$$\begin{aligned}
W(t) &\leq \delta_{18}U(t_1)\exp\Big\{-\delta_{16}(t-t_o)\Big\} + \delta_{17}\int_{t_o}^{t}\phi^{\gamma}(s)d(s)\Big\} \\
&+ \delta_{19}\ell\varepsilon^2,
\end{aligned} \tag{17}$$

where $\delta_{18} = \frac{\delta_{12}}{\delta_{10}}$ and $\delta_{19} = \frac{\delta_{18}\delta_{13}\mathcal{D}_2}{\delta_{16}}$.

If

$$\int_{t_o}^{t}\phi^{\gamma}(s)d(s) < \delta_{16}\delta_{17}^{-1}(t-t_o),$$

then, the exponential index remains negative for all $(t-t_o) \geq 0$. As $t = (t-t_o) \to \infty$ and that $U(t_o)$ is finite in (17), we have that

$$U(t) \leq \delta_{19}\ell\varepsilon^2 \;\; for \;\; any \;\; t.$$

Since U(t) satisfies (9), we get

$$U(t) \leq D_3^{-1}\delta_{19}\ell\varepsilon^2.$$

Also, by (9), we have that

$$|x(t+\tau)-x(t)| + |y(t+\tau)-y(t)| + |z(t+\tau)-z(t)| \leq \left(\frac{3\ell\delta_{19}}{D_3}\right)^{\frac{1}{2}}\varepsilon. \tag{18}$$

Choosing $\ell = \frac{D_3}{3\delta_{19}}$ in (18), we have

$$|x(t+\tau)-x(t)| + |y(t+\tau)-y(t)| + |z(t+\tau)-z(t)| \leq \varepsilon, \tag{19}$$

where τ is chosen to satisfy (12) is relatively dense and hence (19) implies that the solutions $(x(t), y(t), z(t))$ or equivalently $x(t), \dot{x}(t), \ddot{x}(t)$ of (1) are uniformly almost periodic in t.

To show that the solutions are also periodic, we assume that

$$\begin{aligned} q(t+\omega) &= q(t) \\ r(t+\omega, x(t), y(t), z(t)+q(t)) &= r(t, x(t), y(t), z(t)), \end{aligned}$$

for $(x^2+y^2+z^2) \leq D_2$.

Since the perturbation $r(t,x,y,z+q)$ has period ω in t, we replace τ in the definition of $U(t)$ with ω. The terms in the left hand side of (12) is identically zero, thus we may have inequality (19) as

$$|x(t+\omega)-x(t)|+|y(t+\omega)-y(t)|+|z(t+\omega)-z(t)| \leq 0.$$

Thus,

$$|x(t+\omega)-x(t)|+|y(t+\omega)-y(t)|+|z(t+\omega)-z(t)| = 0.$$

which implies that

$$x(t+\omega) = x(t) \;\; and \;\; y(t+\omega) = y(t) \;\; z(t+\omega) = z(t)$$

That is, $x(t), y(t), z(t)$ are periodic in t with period ω.

4 CONCLUSION

Analysis of nonlinear systems literary shows that Lyapunov's theory in Periodic properties of solutions is rarely scarce. The second Lyapunov's method allows to predict the periodic behavior of solutions of sufficiently complicated nonlinear physical system. The solutions of third-order autonomous differential equation (1) are periodic and almost periodic uniformly in x, $\dot{x}$ and $\ddot{x}$ according to Lyapunov's theory if (4) and (5) hold as $t \rightarrow \infty$.

ACKNOWLEDGEMENTS

The author would like to thank the reviewer for his helpful comments and suggestions.

CONFLICTS OF INTERESTS

The author declare no conflict of interest.

AUTHORS' CONTRIBUTIONS

The study was designed and conducted by Olutimo Akinwale. He also wrote the manuscript.

CONSENT

Consent form has been approved by the author.

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