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ORIGINAL RESEARCH

# A Collocation Based Block Multistep Scheme without Predictors for the Numerical Solution Parabolic Partial Differential Equations

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**Abstract:**
**Introduction:** Many life problems often result in differential equations models when formulated mathematically, particularly problems that depend on time and rates which give rise to Partial Differential Equations (PDE).
**Aims:** In this paper, we advance the solution of some Parabolic Partial Dif-ferential Equations (PDE) using a block backward differentiation formula im-plemented in block matrix form without predictors.
**Materials and Methods:** The block backward differentiation formula is devel-oped using the collocation method such that multiple time steps are evaluated simultaneously.
**Results:** A five-point block backward differentiation formula is developed. The stability analysis of the methods reveals that the method is $L_0$ stable.
**Conclusion:** The implementation of some parabolic PDEs shows that the method yields better accuracy than the celebrated Crank–Nicholson's method.
**Keywords:** Collocation, Backward Differentiation Formula, Stability, Crank-Nicholson.

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## 1. INTRODUCTION

Many life problems often result in differential equations when formulated mathematically, particularly problems that depend on time and rates. Some of these problems give rise to Partial Differential Equations (PDE) when using mathematical modeling techniques for describing phenomena in engineering, science, and the business world.

Some known concepts such as Brownian motion, convection, diffusion, conduction, and dispersion are used to simulate applications in telecommunications, economics, biology, engineering, and social sciences. On transformation, using methods of lines, these PDEs are often transformed into systems of Ordinary Differential Equations (ODE) which established some common relationship between them such as change and rates. Some of these ODEs generated fall to the class of stiff problem with initial condition (I.C) whose results are very difficult to arrive at analytically, hence the need for robust numerical methods [1].

Over the years many schemes have been developed to tackle this class of ODEs. Notable among them include Runge-Kutta (RK) methods, Linear Multistep Method, Collocation method, and Hybrid methods.

The general class of the time-dependent PDE is of the form (1) of which the parabolic equation belongs to a special class,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[ r(x,t,u,\frac{\partial u}{\partial x} \right] + f\left( x,t,u,\frac{\partial u}{\partial x} \right) \quad (1)$$

subject to conditions: $u(x,0) = \phi(x)$, $0 \le x \le 1$, $u(0,t) = \psi_0(t)$, $u(1,t) = \psi_1(t)$, $t \ge 0$

Formulating pure PDE problems requires the combination of (1) for modeling some various concepts such as dispersion, convection, conduction in the simulation of application areas in:

- collisions of data packets in a network
- solitons in optical fibres
- stock options
- transport in cellular tissues
- heat transfer
- pollution
- the behaviour of people in a crowd.

This paper discusses the solution of a special case of (1) known as parabolic PDEs which is of the form,

$$\frac{\partial u}{\partial t} = v \frac{\partial^2 u}{\partial x^2}$$

subject to conditions: $u(x,0) = \phi(x)$, $0 \le x \le 1$, $u(0,t) = \psi_0(t)$, $u(1,t) = \psi_1(t)$, $t \ge 0$

In time past, the Crank-Nicholson method has been the celebrated method for solving PDEs, because it transforms the PDEs into systems of linear equations which are solved by abundant existing linear solvers. In trying to improve the accuracy of the results produced by the former, several authors have generated numerical methods for the solution of PDEs by transforming the equations to systems of the first-order ODE using the method of lines, and Higher-Order methods are developed to solve the resulting ODEs [2,3]. Cash [4] derived two new finite difference schemes for parabolic equations. Shampine [3] developed ODE solvers using the method of lines; Diamantakis [5] developed a code called the NUMOL for time-dependent PDEs using Runge-Kutta schemes while Mazzia and Mazzia [6], solved PDEs with High-order transverse schemes. Ramos and Vigo-Aguiar [6] developed a third-order backward differentiation formula with Chebyshev-Gauss-Lobatto quadratures, while Jator [8] developed some fifth-order backward differentiation formula with some other hybrid points. Ngawne and Jator [9], Akinnukawe *et al;* [10] developed some block methods for parabolic partial differential equations. In this paper, a block multistep method is derived via the multistep collocation technique [11, 12, 13, 14, 15, 16]. This method shall be used to generate the numerical solution to some parabolic PDEs.

This paper is structured as follows: The theoretical procedure is presented in section 2 which involves the framework for the transformation of the PDE to ODE using the methods of the line as discussed in Lambert [2] and Schiesser [17]. Section 3 discusses methodology by the derivation of the continuous Backward Differentiation Formula which forms the Block Multistep Method, some stability properties of the block multistep methods, the implementation strategy for the derived methods using Newton's method and we conclude this section by some experimental problems. Finally, we Finally, we give some concluding remarks in section 4.

## 2. MATERIAL AND METHODS

### 2.1 Theoretical procedure

In this section, we present the transformation of a parabolic PDE to systems of ODE. The PDEs shall be discretized using the methods of lines into systems of ODEs and the newly derived methods shall be used to solve the PDE.

Considering the parabolic PDE (2) given as,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \quad , \qquad u(x,0) = \phi(x), \qquad 0 \le x \le 1;$$

$$u(0,t) = \psi_0(t), \;\; u(1,t) = u_1(t), \; t \ge 0 \qquad (2)$$

The solution space involving a rectangular grid with sides parallel to the $x-$axes and $t-$axes, and with good spacing $\Delta x$ and $\Delta t$, when $m\Delta x = 1$ is considered.

A mesh ratio $r = \dfrac{\Delta t}{(\Delta x)^2}$ is specified, denoting $u(i\Delta x, t)$ by ${}^{i}u(t), \;\; i = 1, 2, \cdots, M-1$, with

$$\frac{\partial^2 u(i\Delta x, t)}{\partial x^2} = \frac{\left[{}^{i+1}u(t) - 2\,{}^{i}u(t) + {}^{i-1}u(t)\right]}{(\Delta x)^2} + O\left((\Delta x)^2\right)$$

The solution of (2) is approximated by the solution of the system of the first-order ODE of the form,

$$\left.\begin{aligned} \frac{d\,{}^{i}u(t)}{dt} &= \nu \cdot \frac{\left[{}^{i+1}u(t) - 2\,{}^{i}u(t) + {}^{i-1}u(t)\right]}{(\Delta x)^2} + O\left((\Delta x)^2\right) \\ i &= 1, 2, \cdots, M-1 \\ {}^{i}u(0) &= \phi(i\Delta x) \end{aligned}\right\} \qquad (3)$$

Since ${}^{0}u(t) = \psi_0(t)$ and ${}^{m}u(t) = \psi_1(t)$ are known functions, therefore, (3) reduces to a system of ODE in the $M-1$ unknowns, ${}^{i}u(t) \quad i = 1, 2, \cdots, M-1$ which can be viewed as a tridiagonal matrix form,

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-3} \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = \frac{\nu}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & 0 & & & 0 \\ 0 & 1 & -2 & 1 & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & 1 & -2 & 1 & 0 \\ 0 & & & 0 & 1 & -2 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-3} \\ u_{N-2} \\ u_{N-1} \end{pmatrix} \qquad (4)$$

Where $\Delta x = \dfrac{1}{N+1}$. With the systems of equation (4), the eigenvalues of the matrix are

$$\lambda_i = -\frac{2 + 2\cos(i\pi\Delta x)}{(\Delta x)^2},$$

$$i = 1, 2, \cdots, N$$

Which is in the range $\left(-4(N+1)^2, 0\right)$. Hence for large $N$, the system of the first order (4) becomes very stiff.

## 3. RESULTS AND DISCUSSION

### 3.1 Development of the new method.

In this section, we present the development of a Block Backward Differentiation Formula derived from a continuous multistep scheme via the collocation technique.

Consider an initial value problem of system of ODE,

$$\bar{y} = \bar{f}(x, \bar{y}), \qquad \bar{y}(x_0) = \bar{y}_0 \qquad (5)$$

where $\bar{f}$ satisfies the conditions of existence and uniqueness of the solution. Using the multistep collocation technique to derive the Block Backward Differentiation Formula in form of a block multistep matrix formula [18, 19, 20].

$$A \cdot Y_{n+1} = B \cdot Y_n + h \cdot C \cdot F_{n+1} \qquad (6)$$

with

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kk} \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1k} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2k} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{k1} & b_{k2} & b_{k3} & \cdots & b_{kk} \end{pmatrix},$$

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1k} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2k} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{k1} & c_{k2} & c_{k3} & \cdots & c_{kk} \end{pmatrix}$$

and

$$Y_n = \left[y_{n-k+1}, y_{n-k+2}, \cdots, y_n\right]^T$$

$$Y_{n+1} = \left[y_{n+1_1}, y_{n+2}, \cdots, y_{n+k}\right]^T$$

$$F_{n+1} = \left[f_{n+1_1}, f_{n+2}, \cdots, f_{n+k}\right]^T$$

For a fifth-order method, we set the basis function as,

$$y(x) = \sum_{j=0}^{5} a_j \cdot \left(\frac{x - x_n}{h}\right)^j, \qquad (7)$$

The basis function (7) is interpolated at $x = x_{n+j} \,:\, j = 0, \cdots, 4$ and collocated at $x = x_{n+j} \,:\, j = 5$. This leads to a system of equations in the matrix form,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 0 & 1 & 10 & 75 & 500 & 3125 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ h \cdot f_{n+5} \end{pmatrix}$$

Solving the equations and substituting the coefficients $a_j \,:\, j = 0, 1, 2, \cdots, 5$ in (7), we obtain the continuous multistep formula

$$\begin{aligned} y(x) = {} & \left(1 - \frac{3725}{1644}\left(\frac{x-x_n}{h}\right) + \frac{2015}{1096}\left(\frac{x-x_n}{h}\right)^2 - \frac{2245}{3288}\left(\frac{x-x_n}{h}\right)^3 + \frac{129}{1096}\left(\frac{x-x_n}{h}\right)^4 - \frac{25}{3288}\left(\frac{x-x_n}{h}\right)^5\right) y_n \\ & + \left(\frac{670}{137}\left(\frac{x-x_n}{h}\right) + \frac{5087}{822}\left(\frac{x-x_n}{h}\right)^2 + \frac{4601}{1644}\left(\frac{x-x_n}{h}\right)^3 - \frac{221}{411}\left(\frac{x-x_n}{h}\right)^4 + \frac{61}{1644}\left(\frac{x-x_n}{h}\right)^5\right) y_{n+1} \\ & + \left(-\frac{645}{137}\left(\frac{x-x_n}{h}\right) + \frac{4553}{548}\left(\frac{x-x_n}{h}\right)^2 - \frac{2461}{548}\left(\frac{x-x_n}{h}\right)^3 + \frac{527}{548}\left(\frac{x-x_n}{h}\right)^4 - \frac{39}{548}\left(\frac{x-x_n}{h}\right)^5\right) y_{n+2} \\ & + \left(\frac{1190}{411}\left(\frac{x-x_n}{h}\right) - \frac{1531}{274}\left(\frac{x-x_n}{h}\right)^2 + \frac{5663}{1644}\left(\frac{x-x_n}{h}\right)^3 - \frac{112}{137}\left(\frac{x-x_n}{h}\right)^4 + \frac{107}{1644}\left(\frac{x-x_n}{h}\right)^5\right) y_{n+3} \\ & + \left(-\frac{445}{548}\left(\frac{x-x_n}{h}\right) - \frac{5357}{3288}\left(\frac{x-x_n}{h}\right)^2 - \frac{3517}{3288}\left(\frac{x-x_n}{h}\right)^3 + \frac{907}{3288}\left(\frac{x-x_n}{h}\right)^4 - \frac{77}{3288}\left(\frac{x-x_n}{h}\right)^5\right) y_{n+4} \\ & + h \cdot \left(\frac{12}{137}\left(\frac{x-x_n}{h}\right) + \frac{25}{137}\left(\frac{x-x_n}{h}\right)^2 + \frac{35}{274}\left(\frac{x-x_n}{h}\right)^3 + \frac{5}{137}\left(\frac{x-x_n}{h}\right)^4 + \frac{1}{274}\left(\frac{x-x_n}{h}\right)^5\right) f_{n+5} \end{aligned}$$

Evaluating the continuous multistep formula (8) at $x = x_{n+5}$ and $y'(x)$ at $x = \{x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}\}$, we obtain a block multistep formula in form of (6) with coefficients $a_{ij}$ and $b_{ij}$. $i, j = 1, \cdots, 5$.

$$A = \begin{pmatrix} 1 & -\frac{396}{217} & \frac{183}{217} & -\frac{46}{217} & 0 \\ \frac{71}{39} & 1 & -\frac{127}{39} & \frac{97}{156} & 0 \\ \frac{57}{91} & -\frac{63}{26} & 1 & \frac{321}{364} & 0 \\ -\frac{728}{2501} & \frac{2124}{2501} & -\frac{4008}{2501} & 1 & 0 \\ \frac{75}{137} & -\frac{200}{137} & \frac{300}{137} & -\frac{300}{137} & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{6}{31} \\ 0 & 0 & 0 & 0 & \frac{29}{156} \\ 0 & 0 & 0 & 0 & \frac{31}{364} \\ 0 & 0 & 0 & 0 & -\frac{111}{2501} \\ 0 & 0 & 0 & 0 & \frac{12}{137} \end{pmatrix}$$

$$C = \begin{pmatrix} -\frac{411}{434} & 0 & 0 & 0 & -\frac{9}{434} \\ 0 & -\frac{137}{39} & 0 & 0 & \frac{2}{39} \\ 0 & 0 & \frac{411}{182} & 0 & \frac{9}{182} \\ 0 & 0 & 0 & \frac{1644}{2501} & -\frac{144}{2501} \\ 0 & 0 & 0 & 0 & \frac{60}{137} \end{pmatrix}$$

which shall give the numerical solution to the PDE without predictors.

### 3.2 Stability Properties of the Block Multistep Method

Applying (6) with the coefficient matrix $A$, $B$, and $C$ to the test problem $y' = \lambda y$, we obtain a characteristic equation whose maximum roots give the stability function given as

$$R(z) = \frac{60 + 120z + 105z^2 + 50z^3 + 12z^4}{60 - 180z + 255z^2 - 225z^3 + 137z^4 - 60z^5}. \qquad (9)$$

from the characteristics equation and stability function, we obtain a region of absolute stability $z \in (-\infty, 0) \cup (2.33, \infty) \subset S$ that shows some stiff stability characteristics.

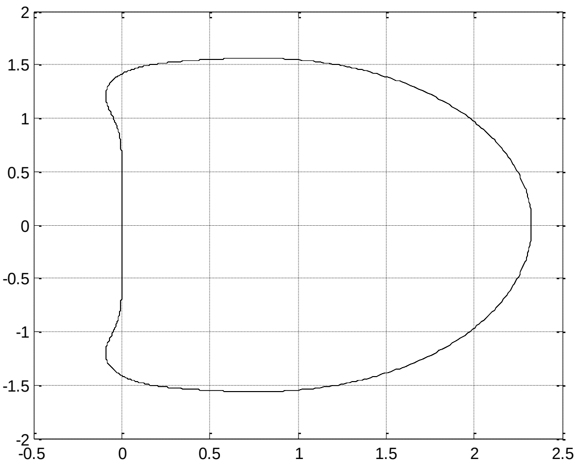


Figure 1: RAS of the Block Multistep Method

Further investigation reveals the $A(\alpha)$ stability properties with $\alpha = 85.56^{\circ}$ $A_0$ and stable with $D = 0.09$. We further investigate the desired property for highly stiff properties called the $L-$stability properties. We take the limit

$$\lim_{z \to -\infty} R(z) = 0$$

This implies that the Block Multistep Method is almost $L-$stable or $L_0$ $-$stable, (for example, see [7,20,21]. These properties reveal that the numerical solutions tend to zero as quickly as possible for very fast decaying solutions.

### 3.2 Implementation Strategy

Since the Block Multistep Method is implicit in nature, Newton's iteration is used to obtain the numerical solution with the Block Multistep Method (6). The Block multistep method

$$A \cdot Y_{n+1} = B \cdot Y_n + h \cdot C \cdot F_{n+1}$$

Is rewritten as

$$F(Y) = A \cdot Y_{n+1} - B \cdot Y_n - h \cdot C \cdot F_{n+1} = 0. \qquad (10)$$

The nonlinear equation which evolves is then solved using the Newton method, which when applied on (10), the numerical solution is obtained with the expression

$$Y_{n+1}^{(s+1)} = Y_n^{(s)} - \left( A - h \cdot C \cdot \frac{\partial F_{n+1}^{(s)}}{\partial Y_n} \right)^{-1} \left( A \cdot Y_{n+1}^{(s)} - B \cdot Y_n^{(s)} - h \cdot C \cdot F_{n+1}^{(s)} \right)$$

### 3.3 Experimental Illustration

In this section, we apply the Block Multistep Method on some parabolic Partial Differential Equations with our new code written in Maple. The errors obtained are generated by $\left| u_{ij} - u(x_i, t_j) \right|$. We compare the numerical solutions obtained with the Crank-Nicholson (CN) method for solving PDE and some other methods in the literature.

**Problem 1**

We consider a simple test parabolic Partial Differential Equation solved in Cash [4] and Jator [8] for $\nu = 1$ which can describe the heat flow on a rod with the cool temperature at the edges given by,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

subject to the Dirichlet conditions $0 \le x \le 1,\ 0 \le t \le 1$, $u(0,t) = u(1,t)$ and $u(x,0) = \sin \pi x$, with an exact solution of

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x).$$

Solving this problem with our Block Multistep Method, we present the numerical results at $t = 1$ for respective values for $\nu = 1$ and $5$ as presented in Table 1.

Table 1: Numerical result for Prob. 1 at $t = 1,\ \Delta x = \Delta t = 0.1$

| $\nu$ | CN | BBDF | Cash [4] | Jator [8] |
|---|---|---|---|---|
| 1 | $3.0 \times 10^{-5}$ | $5.51 \times 10^{-8}$ | $4.5 \times 10^{-6}$ | $1.37 \times 10^{-6}$ |
| 5 | $2.0 \times 10^{-4}$ | $3.56 \times 10^{-11}$ | $2.0 \times 10^{-10}$ | $4.67 \times 10^{-8}$ |

From Table 1, it is clear that the Block Backward Differentiation Formula (BBDF) yields a more accurate result than the result of the Crank-Nicholson computation and outperforms the results obtained by Cash [2] and Jator [7].

**Problem 2**

We consider another parabolic differential equation of the form,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

Subject to the Dirichlet conditions. $0 \le x \le 1$, $0 \le t \le 1$, $u(0,t) = u(1,t)$ and $u(x,0) = \sin \pi x + \sin \varpi \pi x$, with an exact solution of,

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x) + e^{-\varpi^2 \pi^2 t} \sin(\varpi \pi x)$$

It is noted that when $\omega$ increases the transformed equations become very stiff, which implies that only methods that are stable at infinity such as the Block Backward Differentiation Formula derived are capable of coping with such problems. Solving with the Block Backward Differentiation Formula (BBDF), we obtain the following results for $\nu = 1$ and $\varpi = 1, 2, 3, 5, 10$ respectively as presented in Table 2.

Table 2: Numerical result for Problem 2 at $t = 1$

| $\Delta x$ | $\Delta t$ | $\omega$ | $\nu$ | CN | BBDF |
|---|---|---|---|---|---|
| 0.1 | 0.1 | 1 | 1 | $6.20 \times 10^{-5}$ | $1.10 \times 10^{-7}$ |
| 0.1 | 0.1 | 2 | 1 | $3.83 \times 10^{-5}$ | $2.70 \times 10^{-7}$ |
| 0.1 | 0.1 | 3 | 1 | $9.30 \times 10^{-3}$ | $1.30 \times 10^{-4}$ |
| 0.1 | 0.1 | 5 | 1 | $1.80 \times 10^{-1}$ | $5.77 \times 10^{-5}$ |
| 0.1 | 0.1 | 10 | 1 | $6.10 \times 10^{-1}$ | $5.51 \times 10^{-6}$ |

Table 2 shows that this class of numerical methODS IS PROMISING FOR THE NUMERICAL SIMulation of mathematical models arising from physical problems that are IN THE FORM OF PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS.

## 4. CONCLUSION

A block backward differentiation formula has been derived via the multistep collocation technique. The stability analysis of this method reveals that the method is stable at infinity ( $L_0$ -Stable) which makes the method capable of handling stiff equations. The Block Multistep Method is used to solve some parabolic PDEs via transformation to ODE by the method of lines without any predictor. Results obtained have shown that the method yield more accurate result than the Crank-Nicholson method.

## COMPETING INTERESTS

The author declares that there is no conflict of interest.

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