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ORIGINAL RESEARCH

Construction of an Exponentially-Fitted Multiderivative Milne-Simpson Method

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Abstract:
Introduction: Application of classical methods to oscillatory problems is significantly hindered due to the fact that very small step size is required with corresponding decrease in performance, especially in terms of efficiency.
Aim: The aim of this work is to construct a class of two–step exponentially–fitted Milne–Simpson's methods involving first and second derivatives.
Methods: This construction is based on the six-step flow chart described in the literature. Here, a classical multi–derivative Milne–Simpson's method is constructed and fitted exponentially to allow for easy application to oscillatory or periodic problems.
Results: This work extends the classical two-step fourth-order Milne-Simpson's method to involve the second derivative and hence increasing the attainable order of the method, the extended method is also fitted exponentially.
Conclusion: The constructed class of methods is shown to be of order of six (6) and well suited for oscillatory problems.
Keywords: Multi–Derivative, Milne-Simpson, Exponentially–Fitted, Oscillatory, Periodic

All co-authors agreed to have their names listed as authors.

1 INTRODUCTION

Numerical algorithms for solving the initial value problems

$$\begin{aligned} &u^{(n)} = f(t, u, u', u'', \cdots, u^{(n-1)}), \\ &t \in [t_0, T]\,, \;\; u^{(v)}(t_0) = \eta_v,\; v = 0, 1, 2, \cdots, n-1 \end{aligned} \tag{1}$$

whose solution exhibits a pronounced oscillatory behaviour has since the last decade gained a lot of attention. Such problems are often encountered in fields like electronic, astrophysics, mechanics, chemistry and engineering. Several classical methods (methods with only monomials as basis) have been developed by many authors for solving different classes of (1) [1, 2, 3, 4, 5, 6, 7]. However, classical methods may not be well-suited for handling pronounced periodic or oscillatory behaviour, because in order to accurately achieve this, a very small step size would be required with corresponding decrease in performance, especially in terms of efficiency. To overcome this barrier, classical methods have to be adapted in order to efficiently approach the oscillatory behaviour. The adaptation is achieved by replacing some of the highest order monomials of the basis with exponentials or trigonometric [8, 9, 10]. The idea of using exponentially fitted formulae for differential equations was first proposed by authors in [11]. Integration formulae containing free parameters were derived and these parameters were chosen so that a given function exp(q) where *q* is real, satisfies the integration formulae exactly. This was tested on linear multistep method for $k = 1$, however, the authors in [12] derived an A–stable fourth order exponentially fitted formulae based on a linear 2-step formula. Based on this idea, the author in [13], attempted using Multiderivative Linear Multistep Method (MLMM) with *k=1* in the second derivative formulae. In the case of specially adapted methods, particular Runge–Kutta (RK) algorithms have been proposed by several authors [14, 15, 16, 17] in order to solve this class of problems. On the other hand, authors in [18, 19] introduced other exponentially fitted RK (EFRK) methods which integrate exactly first-order systems whose solutions can be expressed as linear combinations of functions of the form
$\{\text{exp}(\lambda t),\ \text{exp}(-\lambda t)\}$ or $\{\text{cos}(\omega t),\ \text{sin}(\omega t)\}$. Recently, the authors in [20] constructed an implicit trigonometrically fitted single-step method having second derivative using trigonometric basis function. In [21], the authors constructed a family of exponentially-fitted one-step Obrechkoff method using the six-step procedure described in [8]. Here, we analyze the construction of a class of two-step exponentially fitted Milne-Simpson's method involving first and second derivatives for solving

$$u' = f(t, u), \;\; t \in [t_0, T]\,, \;\; u(t_0) = u_0 \tag{2}$$

taking into account the six-step flow chart described in [8].

2 METHOD

2.1 Construction of Method

The classical fourth order Milne-Simpson method for solving the first order IVP (2) is given by

$$u_{j+1} = u_{j-1} + \frac{h}{3}\left(f_{j+1} + 4f_j + f_{j-1}\right). \tag{3}$$

In this work, we extend the classical two-step fourth-order Milne-Simpson (3) to involve the second derivative and hence increasing the attainable order of the method, also the extended method is exponentially fitted taking into account the six-step flow chart described in [8]. To begin the construction of our method, we rewrite (3) in a more general way as

$$\begin{aligned} u_{j+1} = a_0 u_{j-1} + h\left(b_0 f_{j+1} + b_1 f_j + b_2 f_{j-1}\right) + \\ h^2\left(c_0 f'_{j+1} + c_1 f'_j + c_2 f'_{j-1}\right) \end{aligned} \tag{4}$$

so as to contain the desired second derivative. Following the six-step procedure described in ([8]), the corresponding linear difference operator $\mathcal{L}[h, \textbf{a}]$ reads

$$\begin{aligned} \mathcal{L}[h, \textbf{a}]u(t) = u(t+h) - a_0 u(t-h) - \\ h((b_0 u'(t+h) + b_1 u'(t) + b_2 u'(t-h)) - \\ h^2((c_0 u''(t+h) + c_1 u''(t) + c_2 u''(t-h)) \end{aligned} \tag{5}$$

where **a** $:= (a_0, b_0, b_1, b_2, c_0, c_1, c_2)$. Applying step II of the six-step procedure we have the system

$$
\begin{aligned}
L_0^*(\mathbf{a}) &= 1 - a_0 = 0 && (6)\\
L_1^*(\mathbf{a}) &= 1 + a_0 - b_0 - b_1 - b_2 = 0 && (7)\\
L_2^*(\mathbf{a}) &= 1 - a_0 - 2b_0 + 2b_2 - 2c_0 - 2c_1 - 2c_2 = 0 && (8)\\
L_3^*(\mathbf{a}) &= 1 + a_0 - 3b_0 - 3b_2 - 6c_0 + 6c_2 = 0 && (9)\\
L_4^*(\mathbf{a}) &= 1 - a_0 - 4b_0 + 4b_2 - 12c_0 - 12c_2 = 0 && (10)\\
L_5^*(\mathbf{a}) &= 1 + a_0 - 5b_0 - 5b_2 - 20c_0 + 20c_2 = 0 && (11)\\
L_6^*(\mathbf{a}) &= 1 - a_0 - 6b_0 + 6b_2 - 30c_0 - 30c_2 = 0 && (12)
\end{aligned}
$$

The algebraic system above is compatible and one finds $M = 7$. The solution of the above system of linear equations will only give the coefficients of the classical method to be adapted. To fit the classical method exponentially, we proceed to step III of the six-step flow chart. Applying step III, we find that

$$
\begin{aligned}
G^{+}(Z, \mathbf{a}) &= (a_0 - 1)\left(-\cosh\left(\sqrt{Z}\right)\right) + (b_2 - b_0)\sqrt{Z}\sinh\left(\sqrt{Z}\right) - \\
&\quad Z\left((c_0 + c_2)\cosh\left(\sqrt{Z}\right) + c_1\right) \\
G^{-}(Z, \mathbf{a}) &= \frac{1}{\sqrt{Z}}\left(\sinh\left(\sqrt{Z}\right)(1 + a_0 + (c_2 - c_0)Z) - \right. \\
&\quad \left. \sqrt{Z}\left((b_0 + b_2)\cosh\left(\sqrt{Z}\right) + b_1\right)\right)
\end{aligned}
\tag{13}
$$

where ω which is the frequency of oscillation is real or imaginary, $z = \omega h = \omega_h$ and $Z = z^2$. (For the trigonometric case, choose $z = \omega h = i\mu h$, i.e $z^2 = -\mu^2 h^2 = Z$.) To implement step IV, Consider the reference set of M functions:

$$\left\{1, t, \cdots, t^K, \exp(\pm\omega t), t\exp(\pm\omega t), \cdots, t^P\exp(\pm\omega t)\right\}$$

with $K + 2P = M - 3$. Since for our method $M = 7$, we have four possibilities:

- $K = 6, P = -1$, the classical case with the set
  $1, t, t^2, t^3, t^4, t^5, t^6$
- $K = 4, P = 0$, the mixed case with the set
  $1, t, t^2, t^3, t^4, \exp(\pm\omega t)$
- $K = 2, P = 1$, the mixed case with the set
  $1, t, t^2, \exp(\pm\omega t), t\exp(\pm\omega t)$
- $K = 0, P = 2$, the pure exponentially fitted case with the set $\exp(\pm\omega t), t\exp(\pm\omega t), t^2\exp(\pm\omega t)$

To get the coefficients of the method for each case, we implement step V by solving the algebraic system

$$L_k^* = 0, \ \ 0 \le k \le K, \quad G^{(p)\pm}(Z, \mathbf{a}) = 0, \ \ 0 \le p \le P$$

The coefficients for each case of the constructed method are:

S1 :: (K,P) = (6,-1)

$$
\begin{aligned}
&a_0 = 1, \quad b_0 = \frac{7}{15}, \quad b_1 = \frac{16}{15}, \\
&b_2 = \frac{7}{15}, \quad c_0 = -\frac{1}{15}, \quad c_1 = 0, \quad c_2 = \frac{1}{15}
\end{aligned}
\tag{14}
$$

as coefficients of the classical two-step second derivative extended Milne-Simpson method.

S2 :: (K,P) = (4,0)

$$\left.\begin{aligned}
a_0 &= 1\\
b_0 &= \frac{6\sqrt{Z}+Z\,\mathsf{sinh}\left(\sqrt{Z}\right)-6\,\mathsf{sinh}\left(\sqrt{Z}\right)}{3\sqrt{Z}\left(\sqrt{Z}\,\mathsf{sinh}\left(\sqrt{Z}\right)-2\,\mathsf{cosh}\left(\sqrt{Z}\right)+2\right)}\\
b_1 &= \frac{4\left(Z\,\mathsf{sinh}\left(\sqrt{Z}\right)+3\,\mathsf{sinh}\left(\sqrt{Z}\right)-3\sqrt{Z}\,\mathsf{cosh}\left(\sqrt{Z}\right)\right)}{3\sqrt{Z}\left(\sqrt{Z}\,\mathsf{sinh}\left(\sqrt{Z}\right)-2\,\mathsf{cosh}\left(\sqrt{Z}\right)+2\right)}\\
b_2 &= \frac{6\sqrt{Z}+Z\,\mathsf{sinh}\left(\sqrt{Z}\right)-6\,\mathsf{sinh}\left(\sqrt{Z}\right)}{3\sqrt{Z}\left(\sqrt{Z}\,\mathsf{sinh}\left(\sqrt{Z}\right)-2\,\mathsf{cosh}\left(\sqrt{Z}\right)+2\right)}\\
c_0 &= \frac{3\,\mathsf{sinh}\left(\sqrt{Z}\right)-\sqrt{Z}\left(\mathsf{cosh}\left(\sqrt{Z}\right)+2\right)}{3\sqrt{Z}\left(\sqrt{Z}\,\mathsf{sinh}\left(\sqrt{Z}\right)-2\,\mathsf{cosh}\left(\sqrt{Z}\right)+2\right)}\\
c_1 &= 0\\
c_2 &= \frac{\sqrt{Z}\left(\mathsf{cosh}\left(\sqrt{Z}\right)+2\right)-3\,\mathsf{sinh}\left(\sqrt{Z}\right)}{3\sqrt{Z}\left(\sqrt{Z}\,\mathsf{sinh}\left(\sqrt{Z}\right)-2\,\mathsf{cosh}\left(\sqrt{Z}\right)+2\right)}
\end{aligned}\right\} \tag{15}$$

S3 :: (K,P) = (2,1)

$$\left.\begin{aligned}
a_0 &= 1\\
b_0 &= \frac{\mathsf{csch}^2\left(\frac{\sqrt{Z}}{2}\right)\left(-2\,\mathsf{sinh}^2\left(\sqrt{Z}\right)+\sqrt{Z}\,\mathsf{sinh}\left(\sqrt{Z}\right)+Z\,\mathsf{cosh}\left(\sqrt{Z}\right)\right)}{2\sqrt{Z}\left(\sqrt{Z}-\mathsf{sinh}\left(\sqrt{Z}\right)\right)}\\
b_1 &= \frac{\left(2Z-4\,\mathsf{sinh}^2\left(\sqrt{Z}\right)+\sqrt{Z}\,\mathsf{sinh}\left(2\sqrt{Z}\right)\right)\mathsf{csch}^2\left(\frac{\sqrt{Z}}{2}\right)}{2\sqrt{Z}\left(\mathsf{sinh}\left(\sqrt{Z}\right)-\sqrt{Z}\right)}\\
b_2 &= \frac{\mathsf{csch}^2\left(\frac{\sqrt{Z}}{2}\right)\left(-2\,\mathsf{sinh}^2\left(\sqrt{Z}\right)+\sqrt{Z}\,\mathsf{sinh}\left(\sqrt{Z}\right)+Z\,\mathsf{cosh}\left(\sqrt{Z}\right)\right)}{2\sqrt{Z}\left(\sqrt{Z}-\mathsf{sinh}\left(\sqrt{Z}\right)\right)}\\
c_0 &= -\frac{\sqrt{Z}+\mathsf{sinh}\left(\sqrt{Z}\right)-Z\,\mathsf{coth}\left(\frac{\sqrt{Z}}{2}\right)}{Z\left(\mathsf{sinh}\left(\sqrt{Z}\right)-\sqrt{Z}\right)}\\
c_1 &= 0\\
c_2 &= \frac{\sqrt{Z}-Z\,\mathsf{coth}\left(\frac{\sqrt{Z}}{2}\right)+\left(\mathsf{cosh}\left(\sqrt{Z}\right)-1\right)\mathsf{coth}\left(\frac{\sqrt{Z}}{2}\right)}{Z\left(\mathsf{sinh}\left(\sqrt{Z}\right)-\sqrt{Z}\right)}
\end{aligned}\right\} \tag{16}$$

S4 :: (K,P) = (0,2)

$$\left.\begin{aligned}
a_0 &= 1\\
b_0 &= \frac{-4\,\mathsf{sinh}^2\left(\sqrt{Z}\right)-2\sqrt{Z}\,\mathsf{sinh}\left(2\sqrt{Z}\right)+2Z\left(\mathsf{cosh}\left(2\sqrt{Z}\right)+3\right)}{Z^{3/2}\left(\mathsf{sinh}\left(2\sqrt{Z}\right)-2\sqrt{Z}\right)}\\
b_1 &= \frac{-4\sqrt{Z}\,\mathsf{sinh}\left(\sqrt{Z}\right)+8Z\,\mathsf{cosh}\left(\sqrt{Z}\right)-4\,\mathsf{sinh}^2\left(\sqrt{Z}\right)\mathsf{cosh}\left(\sqrt{Z}\right)}{Z^{3/2}\left(\sqrt{Z}-\mathsf{sinh}\left(\sqrt{Z}\right)\mathsf{cosh}\left(\sqrt{Z}\right)\right)}\\
b_2 &= \frac{2\left(2\,\mathsf{sinh}^2\left(\sqrt{Z}\right)+\sqrt{Z}\,\mathsf{sinh}\left(2\sqrt{Z}\right)-Z\left(\mathsf{cosh}\left(2\sqrt{Z}\right)+3\right)\right)}{Z^{3/2}\left(2\sqrt{Z}-\mathsf{sinh}\left(2\sqrt{Z}\right)\right)}\\
c_0 &= \frac{Z-2\,\mathsf{sinh}^2\left(\sqrt{Z}\right)+\sqrt{Z}\,\mathsf{sinh}\left(\sqrt{Z}\right)\mathsf{cosh}\left(\sqrt{Z}\right)}{Z^{3/2}\left(\sqrt{Z}-\mathsf{sinh}\left(\sqrt{Z}\right)\mathsf{cosh}\left(\sqrt{Z}\right)\right)}\\
c_1 &= 0\\
c_2 &= \frac{-2Z+4\,\mathsf{sinh}^2\left(\sqrt{Z}\right)-\sqrt{Z}\,\mathsf{sinh}\left(2\sqrt{Z}\right)}{Z^{3/2}\left(2\sqrt{Z}-\mathsf{sinh}\left(2\sqrt{Z}\right)\right)}
\end{aligned}\right\} \tag{17}$$

3 RESULTS

3.1 Error Analysis :: Local Truncation Error (*lte*)

The general expression of the leading term of the local truncation error (*lte*) for an exponentially fitted method with respect to the basis functions

$$\left\{1, t, \cdots, t^K, \mathsf{exp}\,(\pm\omega t), t\,\mathsf{exp}\,(\pm\omega t), \cdots, t^P\,\mathsf{exp}\,(\pm\omega t)\right\} \tag{18}$$

takes the form [8]

$$lte^{EF}(t) = (-1)^{P+1}h^M\frac{\mathcal{L}^*_{K+1}(\mathbf{a}(Z))}{(K+1)!Z^{P+1}}D^{K+1}(D^2-\omega^2)^{P+1}u(t) \tag{19}$$

with *K, P* and *M* satisfying the condition $K + 2P = M - 3$. For the four methods constructed above, one finds the following results:

- S1 :: $(K,P)=(6,-1)$

$$lte_{EF}(t) = \frac{1}{4725}h^7 u^{(7)}(t) \tag{20}$$

- S2 :: $(K,P)=(4,0)$

$$\begin{aligned} lte_{EF}(t) &= -\frac{\left(-8\sqrt{Z}+(Z+15)\sinh\left(\sqrt{Z}\right)-7\sqrt{Z}\cosh\left(\sqrt{Z}\right)\right)}{90Z^{3/2}\left(\sqrt{Z}\sinh\left(\sqrt{Z}\right)-2\cosh\left(\sqrt{Z}\right)+2\right)}h^7 \\ &\quad \times\left(\omega^2 u^{(5)}(t)-u^{(7)}(t)\right) \end{aligned} \tag{21}$$

- S3 :: $(K,P)=(2,1)$

$$\begin{aligned} lte_{EF}(t) &= -\frac{h^7}{12Z^3\left(\sqrt{Z}-\sinh\left(\sqrt{Z}\right)\right)}\operatorname{csch}^2\left(\frac{\sqrt{Z}}{2}\right) \\ &\quad \times\left(2Z^{3/2}-6\sqrt{Z}-(8Z+12)\sinh\left(\sqrt{Z}\right)+\right. \\ &\quad\quad (Z+6)\sinh\left(2\sqrt{Z}\right) \\ &\quad\quad \left.+\,4(Z+3)\sqrt{Z}\cosh\left(\sqrt{Z}\right)-6\sqrt{Z}\cosh\left(2\sqrt{Z}\right)\right) \\ &\quad \times u^{(7)}(t)-2\omega^2u^{(5)}(t)+\omega^4u^{(3)}(t) \end{aligned} \tag{22}$$

- S4 :: $(K,P)=(0,2)$

$$\begin{aligned} lte_{EF}(t) &= \frac{2}{2Z^5-Z^{9/2}\sinh\left(2\sqrt{Z}\right)}h^7 \\ &\quad \times\left(2+6Z+2Z^2-(1+8Z)\cosh\left(\sqrt{Z}\right)\right. \\ &\quad +2(Z-1)\cosh\left(2\sqrt{Z}\right)+\cosh\left(3\sqrt{Z}\right) \\ &\quad \left.+\ 4\sqrt{Z}\sinh\left(\sqrt{Z}\right)-\left(Z^{3/2}+2\sqrt{Z}\right)\left(\sinh\left(2\sqrt{Z}\right)\right)\right) \\ &\quad \times\left(-u^{(7)}(t)+3\omega^2u^{(5)}(t)-3\omega^4u^{(3)}(t)+\omega^6u'(t)\right) \end{aligned} \tag{23}$$

3.2 Convergence and Stability Analysis

Theorem 3.1 (Dahlquist Theorem) *The necessary and sufficient conditions for a linear multistep method to be convergent are that it be consistent and zero-stable*

Dahlquist theorem (3.1) holds also true for EF-based algorithms but, because their coefficients are no longer constants the concepts of consistency and stability have to be adapted.

Definition 3.2 *An exponentially fitted method associated with the fitting space (18) is said to be of exponential order* q*, relative to the frequency* ω *if* q *is the maximum value of* M *such that the algebraic system* $\{\mathcal{L}^*_m(\mathbf{a})=0|m=0,\cdots,M-1\}$ *can be solve.*

Definition 3.3 *A linear multistep method is said to be consistent if it has order* $p\geq 1$

Since $M\geq 1$ for all the derived schemes, the consistency requirement is satisfied. Hence, the derived schemes are all consistent.

Definition 3.4 *A linear* s*-step method is said to be weakly stable if there is more than one simple root of the polynomial equation* $\rho(\xi)=0$ *on the unit circle.*

The stability regards the way how the errors accumulate when the solution is propagated along the interval of interest. The zero-stability refers to the limit case $h\to 0$ but in applications where only significantly non-vanishing steps are used, of course. In [8] the first and second order equations were examined in detail in the exponential fitting context. The idea consists of choosing a differential equation whose analytic solution does not increase indefinitely when $x \to \infty$ and then checking whether the numerical solution conserves this property. For first order equations the test equation is $u' = \lambda u$, $t \geq 0$ with $\Re(\lambda) < 0$. Application of an *s*-step method on the test equation will lead to an *s*-order difference equation whose characteristic equation has *s* roots and the stability properties depend on the magnitude of these roots. For the versions presented above for the Milne-Simpson methods with $\omega_h = \omega h$ and $Z = \omega_h^2$ the second order difference equation is

$$(1 - b_0\bar{h})u_{n+1} - b_1\bar{h}u_n - (1 + b_2\bar{h})y_{n-1} = 0, \quad n = 1, 2, \cdots \tag{24}$$

where $\bar{h} = \lambda h$. The characteristics equation is given by

$$(1 - b_0\bar{h})\xi^2 - b_1\bar{h}\xi - (1 + b_2\bar{h}) = 0 \tag{25}$$

setting $\bar{h} = 0$ in (25), gives the reduced characteristic equation as $\xi^2 - 1 = 0$. The roots are $\xi = \pm 1$ and hence the methods derived are weakly stable. Notice that $\bar{h}$ depends on the test equation but Z on the numerical method, and also that there is no Z dependence in the classical version (14).

4 CONCLUSION

The exponentially–fitted versions of an extended classical Milne-Simpson method have been constructed in this paper. The constructed class of methods are shown to be convergent and weakly stable.

AUTHORS' CONTRIBUTIONS

All authors participated actively in this research work and have read and approved the final manuscript.

CONSENT

Consent form has been approved by all authors.

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