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Tropical Roots and Their Multiplicities on the Subsemigroup of Order-Decreasing and Order-Preserving Full Transformation

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Abstract:
Introduction: Let $\Omega_\upsilon = \{1, 2, 3, --, \upsilon\}$ be ordered finite set. The tropical geometry was utilized to analyze the subsemigroup of order-decreasing and order-preserving full transformation semigroups, denoted as $D_\upsilon \cap O_\upsilon = C_\upsilon$. The elements of classical algebra within C_υ were transformed into tropical polynomials, allowing for the determination of tropical roots and their multiplicities through the tropical curve, which was visualized using GeoGebra.
Aims: The primary aim of this study is to establish a connection between tropical geometry and the algebraic structure of Sub semigroup of Order-Decreasing and Order-Preserving full transformation semigroups
Materials and Methods: The study begins by extracting the elements of the C_υ from T_υ detailing its structure and properties. Elements of C_υ are then converted into tropical polynomials using tropical algebraic operations, specifically tropical addition and multiplication. The study examines these tropical polynomials and computes the multiplicities of their roots using GeoGebra software to illustrate them graphically. Theorems are formulated to describe the properties of the tropicalized elements, with proofs provided to support these results. Furthermore, tropical curves are analyzed to observe the structure of the transformations, and examples are presented to illustrate and confirm the theoretical findings.
Results: The findings reveal that, although the tropical roots differ for each value of υ, the corresponding multiplicities remain the same, and the sum of these multiplicities equals the degree of the associated classical polynomial. Moreover, it was observed that for $1 \leq \upsilon \leq 3$, all multiplicities have a height of 1, whereas for $\upsilon \geq 3$, the height of the multiplicities is 2. Additionally, for $\upsilon = 2$, the multiplicities follow the pattern $(1, \upsilon - 2, 1)$.
Conclusion: This work offers insightful results and lays a solid foundation for further exploration in the area of tropical geometry and transformation semigroups, especially for those interested in the interplay between algebraic and tropical structures.

Keywords: Semigroup, Full Transformation, Subsemigroup of Order-Decreasing and Order-Preserving, Tropical polynomial, Tropical curve, Multiplicity and Height.

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All co-authors agreed to have their names listed as authors.

1. INTRODUCTION

A semigroup is an algebraic structure consisting of a non-empty set S equipped with an associative binary operation. That is, for all elements $a, b \in S$, the following condition holds: $a.(b.c) = (a.b).c$

A transformation semigroup is a collection of transformations (i.e., functions from a set to itself) that is closed under function composition. If this collection includes the identity function, it is called a monoid. Transformation semigroups play a fundamental role in theoretical computer science and other areas of mathematics. There are three Transformation semigroups introduced by Howie[6] and defined as below

Let $\Omega_v = \{1, 2, 3, --, v\}$ be a finite, ordered set. A transformation $\alpha: Dom(\alpha) \to Im(\alpha) \subseteq \Omega_v$ is said to be: Full, if $Dom(\alpha) = \Omega_v$ denoted by T_v. Partial, if $Dom(\alpha) \subseteq X$, denoted by P_v and injective partial, if is injective and partial, denoted by I_v .

A subsemigroup of a semigroup S is a non-empty subset $T \subseteq S$ that is closed under the binary operation of S. In other words, T itself forms a semigroup under the same operation. Tropical geometry is not merely a recreational curiosity for mathematicians; rather, it serves as a powerful tool for studying degenerations of classical algebraic structures.

The tropical world can often be viewed as a limit or simplification of the classical world, where many essential properties are preserved. Thus, a tropical statement frequently has a corresponding classical analogue. One of the main advantages of tropical mathematics is that tropical objects are piece-wise linear, making them more tractable than their classical counterparts. Prior to the widespread adoption of the term tropical algebra, this field was commonly known as max-plus, algebra.

A tropical polynomial expression of the form $P(x) = \sum_{i=0}^{v} a_i x^i$ induces a tropical polynomial function, denoted by P on the tropical semiring T;

$$P: \mathrm{T} \longrightarrow \mathrm{T} \quad (1)$$

In mathematics, tropical geometry studies polynomials by examining their geometric properties. Despite its unconventional structure, it satisfies fundamental geometric properties, making it a useful tool for various mathematical disciplines.

The set of tropical numbers is defined as $\mathrm{T} = \mathbb{R} \cup \{-\infty\}$ endowed with the operations called tropical addition and multiplication:

$$\text{"}\mu \oplus \sigma\text{"} = \max\{\mu, \sigma\} \quad (2)$$

$$\text{"}\mu \otimes \sigma\text{"} = \mu \oplus \sigma \quad (3)$$

with the usual conventions$\forall \mu \in \mathrm{T}$, "$\mu \oplus (-\infty)$"=max $(\mu, -\infty) = \mu$ and

$$\text{"}\mu \otimes (-\infty)\text{"} = \mu \oplus (-\infty) = -\infty \quad (4)$$

Unlike classical arithmetic, tropical addition lacks an additive inverse, meaning tropical numbers form a semi-field rather than a full field. Notably, in tropical operations $2\mu \neq$ "$\mu \oplus \mu$" but "2μ" $= \mu \oplus 2$ and "0μ" $= \mu$ but not equal to 0. For example, $3 \oplus 5 = 5$ *and* $3 \otimes 5 = 8$.

Despite the growing interest in tropical geometry see [Bakare [2], Brugalle et. al.[3], Brugalle [4],ibrahim et. al [7], Itemberg et. al[8],Katz [9], Maclagan[10]], its applications in algebraic structures and semigroup theory see [Brugalle and Shaw [5], Usamot et. all[11]], there has been limited investigation into its application to transformation semigroups particularly Sub semigroup of Order-Decreasing and Order-Preserving Full Transformation Semigroups (C_v). However, Umar [12] examined the tropicalization of idempotent elements in full transformation semigroups. Recent findings on tropical geometry and transformation semigroups can be found in [Bakare [1], Umar[6]]. This work connects tropical geometry with semigroup theory, specifically through the study of tropical polynomials and their multiplicities within the semigroup of C_υ. The elements in C_υ are transformed into tropical polynomials, and we established Theorems to describe their properties. Relevant examples are provided to validate the results.

2. METHODOLOGY

The study begins by extracting the elements of the C_υ from T_υ detailing its structure and properties. Elements of C_υ are then converted into tropical polynomials using tropical algebraic operations, specifically tropical addition and multiplication. The study examines these tropical polynomials and computes the multiplicities of their roots using GeoGebra software to illustrate them graphically. Theorems are formulated to describe the properties of the tropicalized elements, with proofs provided to support these results. Furthermore, tropical curves are analyzed to observe the structure of the transformations, and examples are presented to illustrate and confirm the theoretical findings.

Definition 2.1 Brugalle et. al **[3]:** A tropical number τ is a root of a Polynomial $P(x)$ in one variable in which the points x_0 on the graph $P(x)$ has a corner at x_0 for $-\infty \leq \tau \leq \infty$.

Definition 2.2 Brugalle et. al **[3]:** The Multiplicity of a tropical root r denoted by $M(\tau)$ defined as $(\tau_1) = |\Omega_1 - \Omega_2|$

$M(\tau_n) = \sum_{i=1}^{\upsilon} |\Omega_i - \Omega_{i+1}|$ where Ω_i are the slopes of the lines in the tropical curve of $P(x)$ intersecting above τ.

Proposition 2.3 Brugalle et. al **[3]:** The tropical semi-field is algebraically closed. That is, every tropical polynomial of degree $d > 0$ has exactly d roots when counted with multiplicities.

Definition 2.4 Umar[12]: A transformation $\alpha \epsilon C_\upsilon$ is called an order-decreasing transformation if, $\forall\, \mu \epsilon Dom(\alpha)$, we have $\alpha(\mu) \leq \mu$. The set of such transformations forms the order-decreasing transformation semigroup, denoted by D_υ .

Definition 2.5 Umar [12]: A transformation $\alpha \epsilon C_\upsilon$ is called an order-preserving transformation if, $\forall\, \mu, \sigma \in Dom(\alpha)$, with $\mu \leq \sigma$, it holds that $\alpha(\mu) \leq \alpha(\sigma)$. The set of such transformations form the order-preserving transformation semigroup, denoted by O_υ

3. RESULTS AND DISCUSSION

Theorem 3.1: Let $S = C_\upsilon$ be the semigroup of transformations that are both

order-decreasing and order-preserving. Then, every element in C_υ has the same multiplicity

for each υ.

Proof: Let $S = C_\upsilon$ and consider two elements: $\alpha, \beta \epsilon S$ such that $\alpha = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 - - - \psi_\upsilon \\ \varsigma_1 & \varsigma_2 & \varsigma_3 - - - \varsigma_\upsilon \end{pmatrix}$

and $\beta = \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 - - - \omega_\upsilon \\ \tau_1 & \tau_2 & \tau_3 - - - \tau_\upsilon \end{pmatrix}$ for $\psi_\upsilon = \omega_\upsilon \in dom(\alpha, \beta)$ and $\tau_\upsilon \neq \varsigma_\upsilon \in Im(\alpha, \beta)$.

Consider the polynomials defined from $\alpha\ and\ \beta$ i.e

$\alpha(x) = \psi_i x^\upsilon + \psi_{i+1} x^{\upsilon-1} + \psi_{i+2} x^{\upsilon-2} + \cdots + \psi_{i+k} x^{\upsilon-k} - \varsigma_i x^{\upsilon-1} + \varsigma_{i+1} x^{\upsilon-2} - \cdots + \varsigma_{i+k} x^{\upsilon-k}$

and

$\beta(x) = \omega_i x^{\upsilon} + \omega_{i+1} x^{\upsilon-1} + \omega_{i+2} x^{\upsilon-2} + \cdots + \omega_{i+k} x^{\upsilon-k} - \tau_i x^{\upsilon-1} - \tau_{i+1} x^{\upsilon-2} + \cdots + \tau_{i+k} x^{\upsilon-k}$

Their corresponding tropical polynomial forms are;

$\top\ \alpha(x) = "\psi_i x^{\upsilon} + \psi_{i+1} x^{\upsilon-1} + \psi_{i+2} x^{\upsilon-2} + \cdots + \psi_{i+k} x^{\upsilon-k} - \varsigma_i x^{\upsilon-1} + \varsigma_{i+1} x^{\upsilon-2} - \cdots + \varsigma_{i+k} x^{\upsilon-k}"$

$\top\ \beta(x) = "\omega_i x^{\upsilon} + \omega_{i+1} x^{\upsilon-1} + \omega_{i+2} x^{\upsilon-2} + \cdots + \omega_{i+k} x^{\upsilon-k} - \tau_i x^{\upsilon-1} - \tau_{i+1} x^{\upsilon-2} + \cdots + \tau_{i+k} x^{\upsilon-k}"$

which, using tropical operations, correspond to the expressions:

$max\{\psi_i + \upsilon x, \psi_{i+1} + (\upsilon-1)x, \ldots, \psi_{i+k} + (\upsilon-k)x, \varsigma_i - (\upsilon-1)x, \varsigma_{i+1} + (\upsilon-2)x, \varsigma_{i+k} + (\upsilon-k)x\}$

$max\{\omega_i + \upsilon x, \omega_{i+1} + (\upsilon-1)x, \ldots, \omega_{i+k} + (\upsilon-k)x, \tau_i - (\upsilon-1)x, \tau_{i+1} - (\upsilon-2)x,\ \tau_{i+k} + (\upsilon-k)x\}.$

By analyzing the tropical curves associated with $T_{\alpha(x)}$ and $T_{\beta(x)}$ for various values of

x, we observe that although these functions may have different tropical roots, they exhibit

the same multiplicities at those roots. i.e, $|MT_{\alpha(x)}| = |MT_{\beta(x)}|$ ∎

Theorem 3.2: Every transformation in the semigroup, for $1 \leq \upsilon \leq 3$, with difference in tropical root has multiplicity of height 1

Proof. Let us consider transformations in C_{υ} for small values of υ:

Case $\upsilon = 1$

$\alpha = \begin{bmatrix} a_1 \\ g_1 \end{bmatrix} \Rightarrow clasical\ polynomial: f(x) = a_1 x + g_1 \Rightarrow Tropical\ form: T(x) = max(a_1 + x, g_1)$

Case $\upsilon = 2$ $\alpha = \begin{bmatrix} a_1 & a_2 \\ g_1 & g_2 \end{bmatrix} \Rightarrow clasical\ polynomial:\ f(x) = a_1 x^2 + a_2 x + g_1 x + g_2 \Rightarrow$

$Tropical\ form: T(x) = max(a_1 + 2x, a_2 + x, g_1 + x, g_2)$

Case $\upsilon = 3$ $\alpha = \begin{bmatrix} a_1 & a_2 & a_3 \\ g_1 & g_2 & g_3 \end{bmatrix} \Rightarrow clasical\ polynomial:$

$f(x) = a_1 x^3 + a_2 x^2 + a_3 x + g_1 x^2 + g_2 x + g_3 \Rightarrow$

$Tropical\ form: T(x) = max(a_1 + 3x, a_2 + 2x, a_3 + x, g_1 + x, g_2)$

In each of these cases, the tropical polynomial produces a piecewise linear function whose graph has only one corner (i.e., one tropical root), and the change in slope at that corner is 1. Therefore, the multiplicity at the tropical root is: $\mathrm{MT}(x) = 1$

This confirms that for all n such that $1 \leq \upsilon \leq 3$, every transformation in C_{υ} yields a tropical polynomial with multiplicity of height 1.

Theorem 3.3: For all $\upsilon \geq 4$ with difference tropical root, the multiplicities in the semigroup C_{υ} have a uniform height of 2

Proof. This result follows directly from Theorem 2, which established the multiplicity height for $v \leq 3$. For $v \geq 4$, analysis of the tropical polynomials associated with elements of C_v reveals that the change in slope at the tropical roots consistently results in multiplicities of height 2. In particular:

$$\mathrm{MTC}_4 = \mathrm{MTC}_5 = \cdots = 2$$

Thus, for all $v \geq 4$, the tropical multiplicity in C_v satisfies: $\mathrm{MTC}_v = 2$

Examples of Tropical Roots and Multiplicities in C_v

In this section, we examine selected elements from the semigroup C_v for values of $v = 1$ to

$v = 4$. For each case, the corresponding classical polynomial and tropical polynomial are constructed, and the tropical root along with its multiplicity is computed

Table 1. **Tropical Root and Multiplicity for C_1**

C_1	Classical polynomial	Tropical polynomial	Root	Multiplicity
$\begin{pmatrix}1\\1\end{pmatrix}$	$(x+1)$	Max(x,1)	1	1

Table 2. **Tropical Root and Multiplicity for C_2**

C_2	Classical polynomial	Tropical polynomial	Roots	Multiplicities
$\begin{pmatrix}1 & 2\\1 & 1\end{pmatrix}$	(x^2+3x+1)	$\max\{2x, x+3, 1\}$	$-2 and\ 3$	1,1
$\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$	(x^2+3x+2)	$\max\{2x, x+3, 2\}$	$-1 and\ 3$	1,1

Table 3. **Tropical Root and Multiplicity for C_3**

C_3	Classical polynomial	Tropical polynomial	Roots	Multiplicities
$\begin{pmatrix}1 & 2 & 3\\1 & 1 & 1\end{pmatrix}$	x^3+3x^2+4x+1	$\max\{3x, 2x+3, x+4, 1\}$	$-3, 1 and\ 3$	1,1,1
$\begin{pmatrix}1 & 2 & 3\\1 & 1 & 2\end{pmatrix}$	x^3+3x^2+4x+2	$\max\{3x, 2x+3, x+4, 2\}$	$-2, 1 and\ 3$	1,1,1
$\begin{pmatrix}1 & 2 & 3\\1 & 1 & 3\end{pmatrix}$	x^3+3x^2+4x+3	$\max\{3x, 2x+3, x+4, 3\}$	$-1, 1 and\ 3$	1,1,1
$\begin{pmatrix}1 & 2 & 3\\1 & 2 & 2\end{pmatrix}$	x^3+3x^2+5x+2	$\max\{3x, 2x+3, x+5, 2\}$	$-3, 2 and\ 3$	1,1,1
$\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}$	x^3+3x^2+5x+3	$\max\{3x, 2x+3, x+5, 2\}$	$-2, 2 and\ 3$	1,1,1

Table 4. Tropical Root and Multiplicity for C_4

C_4	Classical polynomial	Tropical polynomial	Roots	Multiplicities
$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 1 & 1 & 1\end{pmatrix}$	$x^4 + 3x^3 + 4x^2 + 5x + 1$	$\max\{4x, 3x + 3, 2x + 4, x + 5, 1\}$	$-4,1$ *and* 3	1,2,1
$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 1 & 2 & 2\end{pmatrix}$	$x^4 + 3x^3 + 4x^2 + 6x + 2$	$\max\{4x, 3x + 3, 2x + 4, x + 6, 2\}$	$-4,1.5$ *and* 3	1,2,1
$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 1 & 2 & 3\end{pmatrix}$	$x^4 + 3x^3 + 4x^2 + 6x + 3$	$\max\{4x, 3x + 3, 2x + 4, x + 6, 3\}$	$-3,1.5$ *and* 3	1,2,1

Table 5. Tropical Root and Multiplicity for C_5

C_5	Classical polynomial	Tropical polynomial	Roots	Multiplicities
$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 1 & 1 & 1 & 1\end{pmatrix}$	$x^5 + 3x^4 + 4x^3 + 5x^2 + 6x + 1$	$\max(5x, 4x + 3, 3x + 4, 2x + 5, x + 6, 1)$	$-3,1$ *and* 3	1,3,1
$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 1 & 1 & 1 & 5\end{pmatrix}$	$x^5 + 3x^4 + 4x^3 + 5x^2 + 6x + 5$	$\max(5x, 4x + 3, 3x + 4, 2x + 5, x + 6, 5)$	$-1,1$ *and* 3	1,3,1
$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 1 & 1 & 2 & 2\end{pmatrix}$	$x^5 + 3x^4 + 4x^3 + 5x^2 + 7x + 2$	$\max(5x, 4x + 3, 3x + 4, 2x + 5, x + 7, 2)$	$-5,1.3$ *and* 3	1,3,1

Graphical Illustration

The following figures demonstrate how the tropical roots and their associated multiplicities were obtained with the aid of Geogebra.

Example 1: Let $\alpha, \beta \in C_3$, then consider the transformations $\alpha = (1), (2\ \ 1], (3)$ and $\beta = (1), (2)(\ 3\ 2]$ with polynomial functions;

$$\alpha(x) = x^3 + 3x^2 + 4x + 3$$

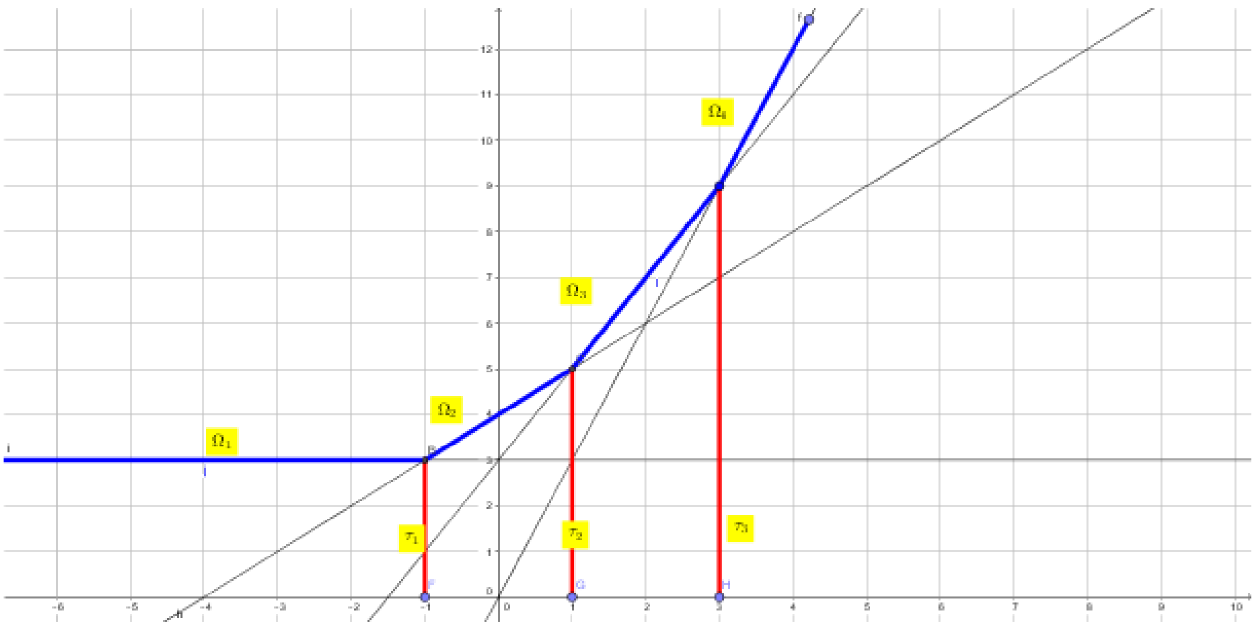
$$\beta(x) = x^3 + 3x^2 + 5x + 2$$

By the virtue of (2) and (3) we have; $T\alpha(x) = max\{3x, 2x + 3, x + 4, 3\}$

$$T\beta(x) = max\{3x, 2x + 3, x + 5, 2\}$$

Graphically,

Tropical curve of $\alpha(x)$



GRAPH I

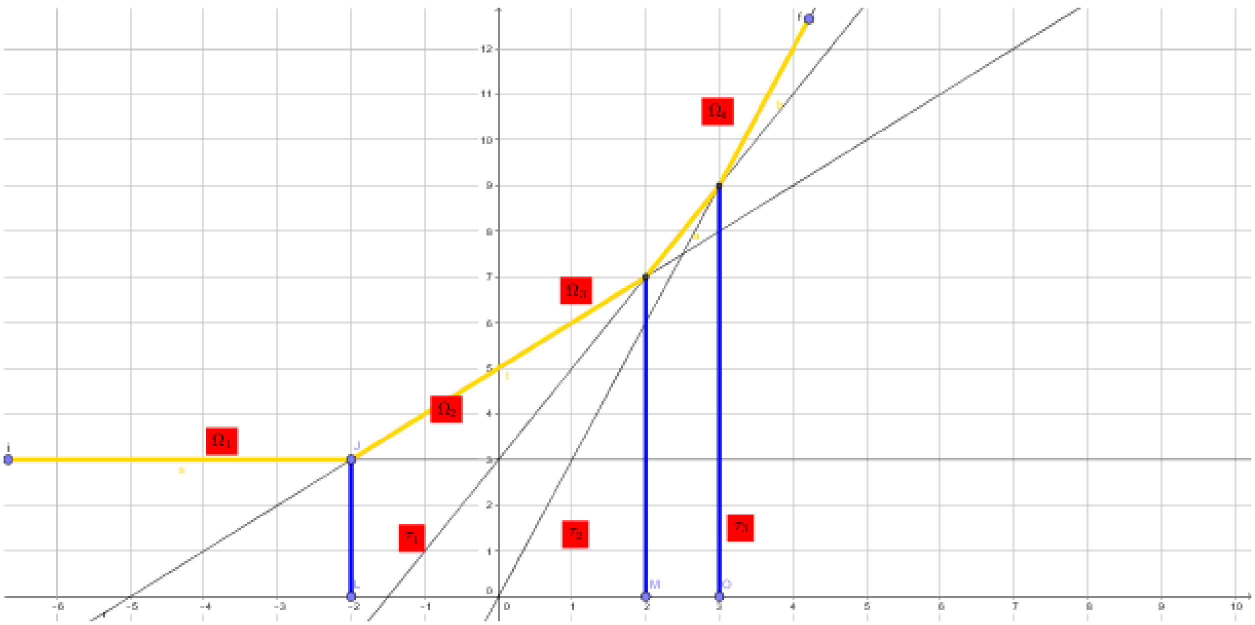
Fig. 1. Example of Tropical Root and Multiplicity for C_3

From Graph I, the roots are; $\tau_1 = -1, \tau_2 = 1$ and $\tau_3 = 3$ with slopes $\Omega_1 = 0, \Omega_2 = 1, \Omega_3 = 2$ and $\Omega_4 = 3$ having multiplicities

$M(\tau_1) = |\Omega_1 - \Omega_2| = 1$, $M(\tau_2) = |\Omega_2 - \Omega_3| = 1$ and $M(\tau_3) = |\Omega_3 - m_4| = 1$

Thus, the multiplicity of $T_{\alpha(x)} = (1\ \ 1\ \ 1) \Rightarrow |MT_{\alpha(x)}| = 1$

Tropical curve of $\boldsymbol{\beta(x)}$



GRAPH II

Fig. 2. Example of Tropical Root and Multiplicity for C_3

From Graph II, the roots are;$\tau_1 = -2, \tau_2 = 2$ and $\tau_3 = 3$ with slopes $\Omega_1 = 0, \Omega_2 = 1, \Omega_3 = 2$ and $\Omega_4 = 3$ having multiplicities

$M(\tau_1) = |\Omega_1 - \Omega_2| = 1$, $M(\tau_2) = |\Omega_2 - \Omega_3| = 1$ and $M(\tau_3) = |\Omega_3 - \Omega_4| = 1$

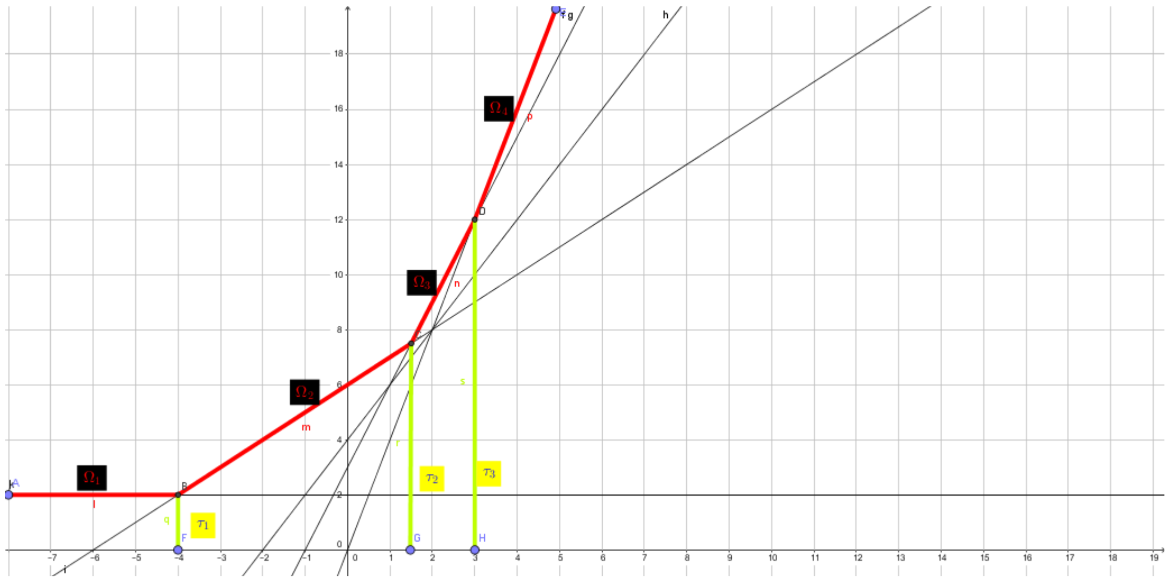
Thus, the multiplicity of $T_{\alpha(x)} = (\,1\;\;1\;\;1) \Rightarrow |MT_{\alpha(x)}| = 1$

Example 2: Let $\xi \,\epsilon\, C_4$, then consider the transformations $\alpha = (1), (2\;\;1], (3\;\;2](4\;2]$ with polynomial functions; $\xi(x) = x^4 + 3x^3 + 4x^2 + 6x + 2$

By the virtue of (2) and (3) we have; $\mathrm{T}\xi(x) = \max\{4x, 3x + 3, 2x + 4, x + 6, 2\}$

Graphically,

Tropical curve of $\xi(x)$



GRAPH III

Fig. 3. Example of Tropical Root and Multiplicity for C$_4$

From Graph III, the roots are; $\tau_1 = -4, \tau_2 = 1.5$ and $\tau_3 = 3$ with slopes $\Omega_1 = 0, \Omega_2 = 1, \Omega_3 = 3$ and $\Omega_4 = 4$ having multiplicities

$M(\tau_1) = |\Omega_1 - \Omega_2| = 1$, $M(\tau_2) = |\Omega_2 - \Omega_3| = 2$ and $M(\tau_3) = |\Omega_3 - \Omega_4| = 1$

Thus, the multiplicity of $T_{\alpha(x)} = (1 \;\; 2 \;\; 1) \Rightarrow |MT_{\alpha(x)}| = 2$

Example 3: Let $\psi \epsilon\, C_4$, then consider the transformations $\psi = (1), (5), (2 \;\; 1], (3 \;\; 1](4\,1]$ with polynomial functions; $\psi(x) = x^5 + 3x^4 + 4x^3 + 5x^2 + 6x + 5$

By the virtue of (2) and (3) we have; $\mathrm{T}\psi(x) = \max\,(5x, 4x + 3, 3x + 4, 2x + 5, x + 6{,}5)$

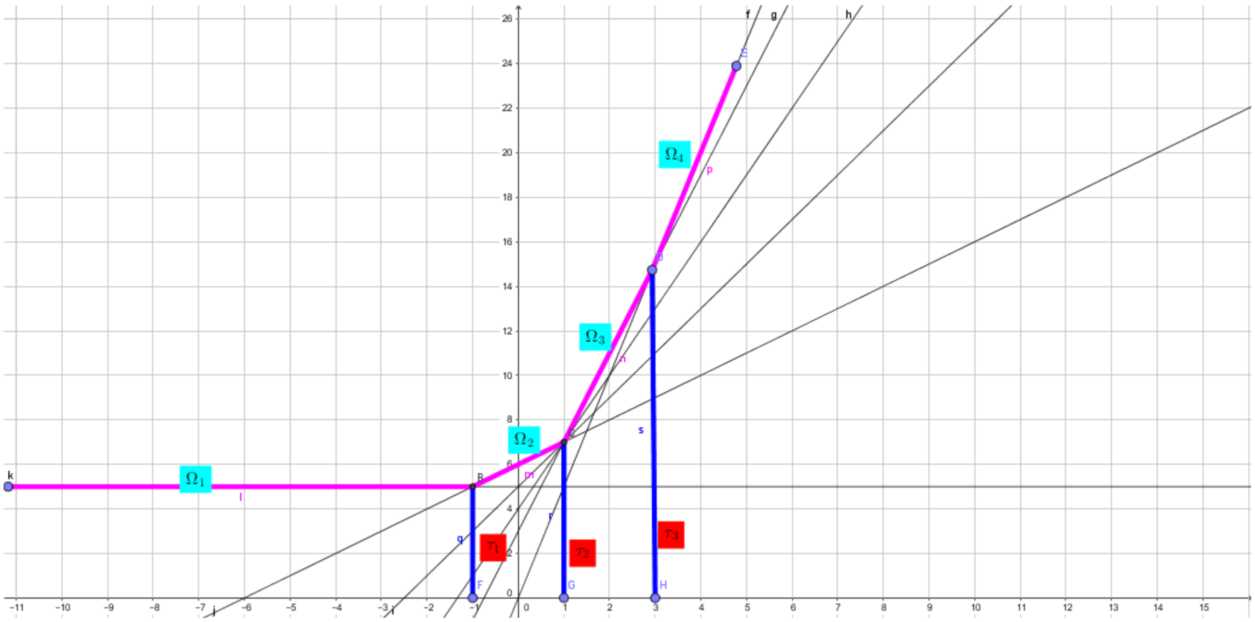
Graphically,

From Graph IV, the roots are; $\tau_1 = -4, \tau_2 = 1$ and $\tau_3 = 3$ with slopes $\Omega_1 = 0, \Omega_2 = 1, \Omega_3 = 4$ and $\Omega_4 = 5$ having multiplicities

$M(\tau_1) = |\Omega_1 - \Omega_2| = 1$, $M(\tau_2) = |\Omega_2 - \Omega_3| = 3$ and $M(\tau_3) = |\Omega_3 - \Omega_4| = 1$

Thus, the multiplicity of $T_{\alpha(x)} = (1 \;\; 3 \;\; 1) \Rightarrow |MT_{\alpha(x)}| = 2$

Tropical curve of $\psi(x)$



GRAPH IV

Fig. 4. Example of Tropical Root and Multiplicity for C_5

4. CONCLUSION

In this study, we examined elements of the full transformation semigroup that are both order decreasing and order-preserving. Our investigation focused on cases where $v = 1, 2, 3, 4, 5$. The findings reveal that, although the tropical roots differ for each value of v, the corresponding multiplicities remain the same, and the sum of these multiplicities equals the degree of the associated classical polynomial.

Moreover, it was observed that for $1 \leq v \leq 3$, all multiplicities have a height of 1, whereas for $v \geq 3$, the height of the multiplicities is 2. Additionally, for $v = 2$, the multiplicities follow the pattern $(1, v - 2, 1)$.

This work offers insightful results and lays a solid foundation for further exploration in the area of tropical geometry and transformation semigroups, especially for those interested in the interplay between algebraic and tropical structures.

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Not applicable

COMPETING INTERESTS

The authors declare that they have no competing interests.

AUTHORS' CONTRIBUTIONS

"RGI" led the conceptual framework of the study by identifying and extracting the elements of $\mathbf{C}_v$ from $\mathbf{T}_v$, "NGB" transform the classical elements into tropical polynomials through tropical algebraic operations, "IFU" assisted in the development and formal proof of theorems relating to the tropicalized elements, "OIS" focused on the application of geogebra software for computing and graphically illustrating the multiplicities of the roots of tropical polynomials, “SAA” contributed to the analysis of tropical curves and the structural behavior of transformations, and “DAA” coordinated the integration of the contributions into a cohesive narrative and ensured clarity, consistency, and academic standards in the manuscript. “All authors read and approved the final manuscript.”

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